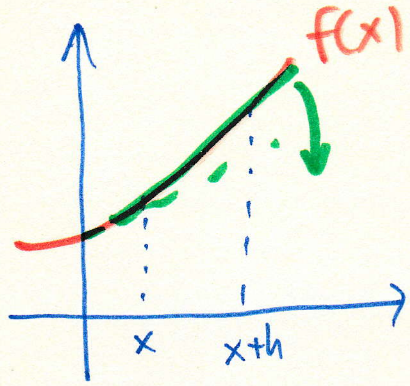
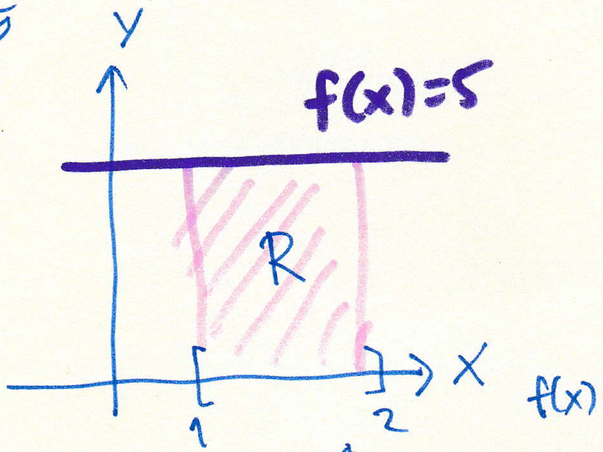
Calculus Derivative



Ex $f(x) = 5$

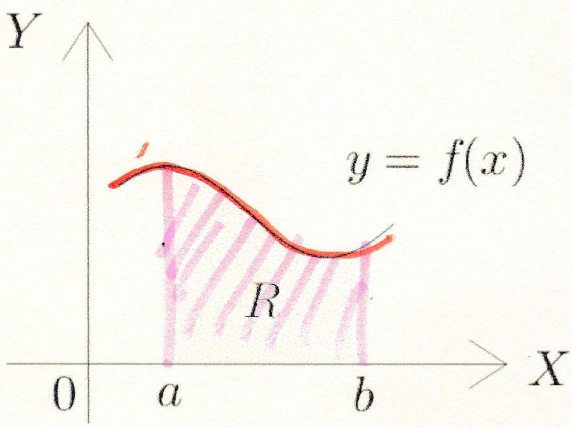


พ.ท.บริเวณ R คือ $\underbrace{1}_{\text{ความกว้าง}} \times \underbrace{5}_{\text{ความสูง}} = 5$ ตร.หน่วย

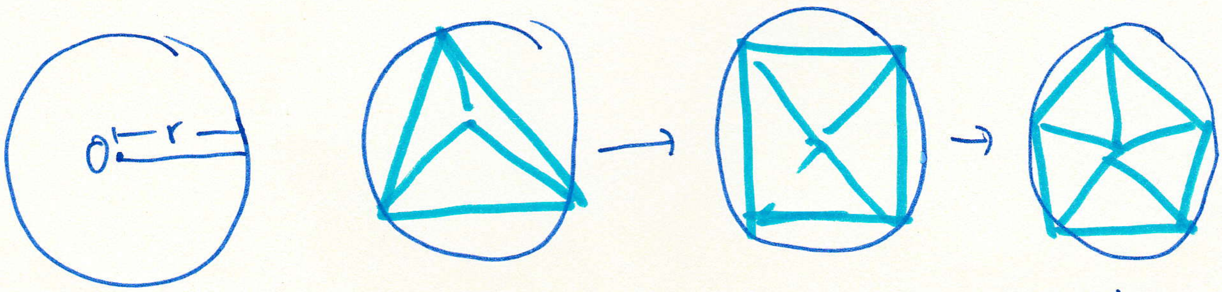
5 Integration

5.1 An Overview of the Area Problem

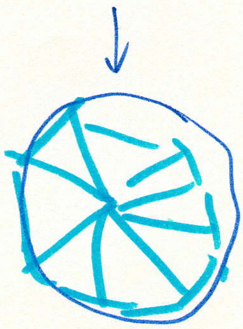
Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 5.1).



Archimedes' method of exhaustion



พ.ท. วงกลม $= \pi r^2$



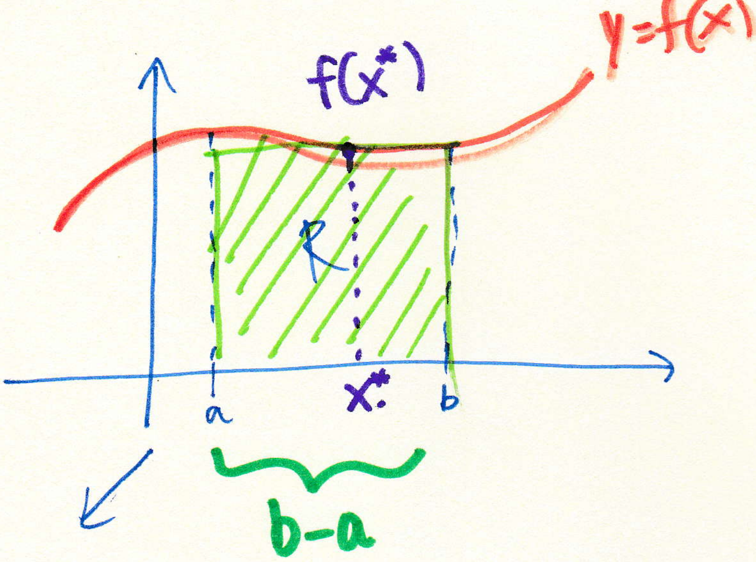
ให้ A_n แทนพื้นที่ของรูป n เหลี่ยมแนบในวงกลม

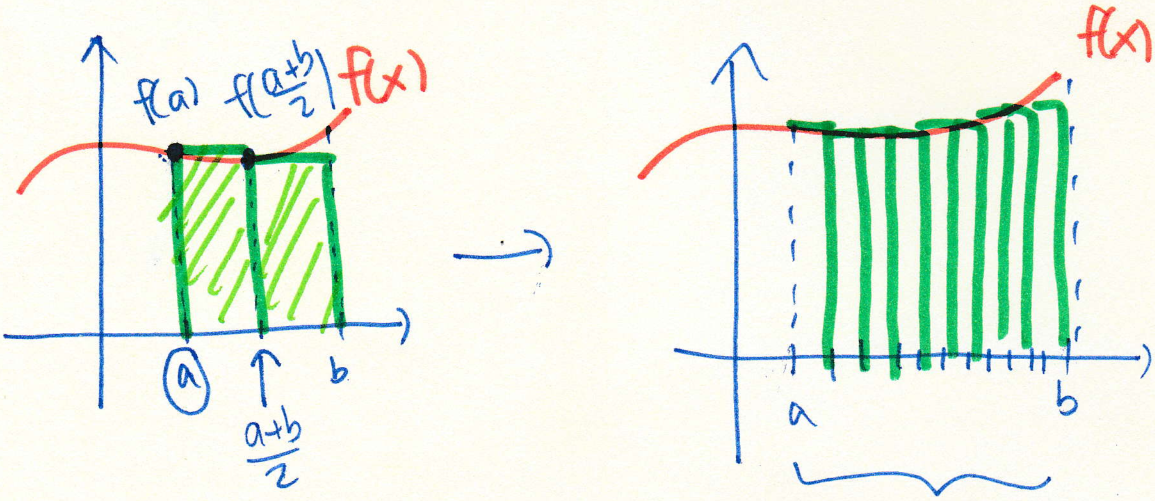
ถ้า A แทน พ.ท. ของวงกลม จะได้ว่า

$$A = \lim_{n \to \infty} A_n$$

$f(x) \geq 0$

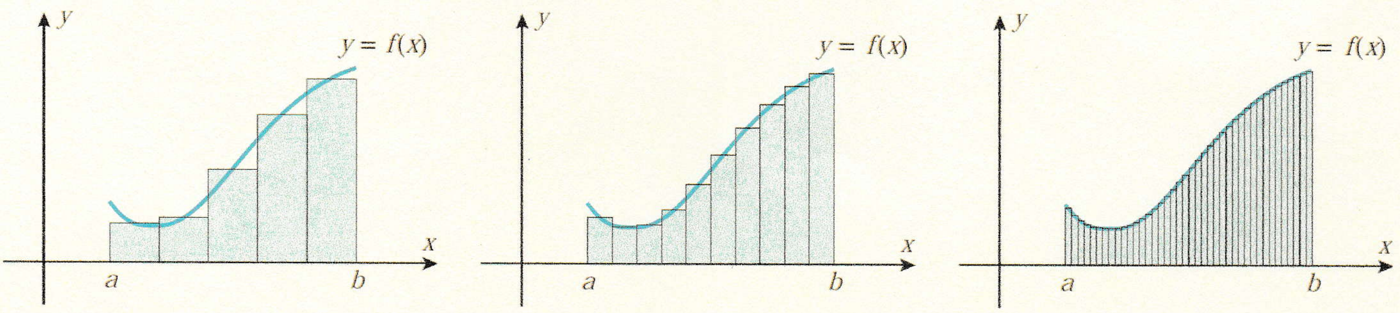
Let $y = f(x)$ be a function over the interval $[a, b]$. We consider the rectangle method for computing the area A under the curve $f(x)$ on the interval $[a, b]$.





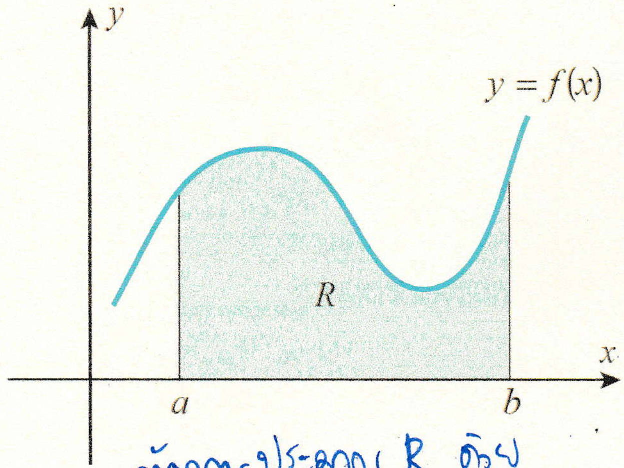
That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n\to\infty} A_n$$

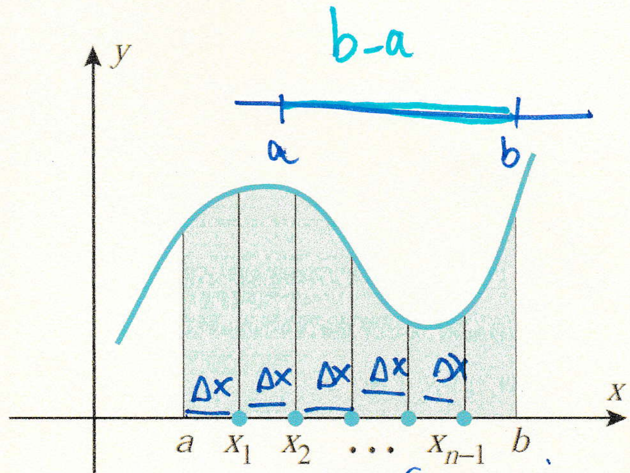


DEFINITION 5.1 (Area Under a Curve) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

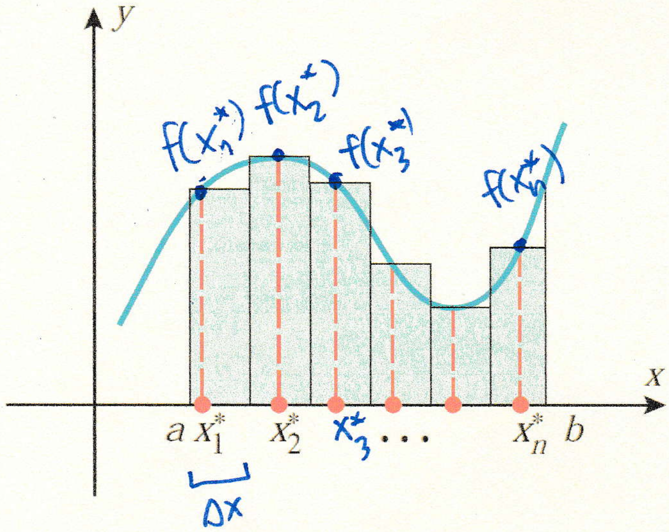
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

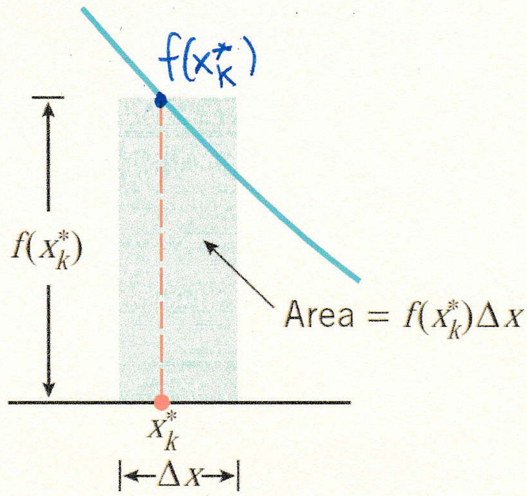


ต้องการประมาณ R ด้วย ▭ n ช่อง



แบ่ง $[a,b]$ ออกเป็น n ช่อง
สมมติว่าแบ่งให้แต่ละช่วงยาวเท่ากัน
$\therefore$ ความกว้างช่วง : $\Delta x = \frac{b-a}{n}$



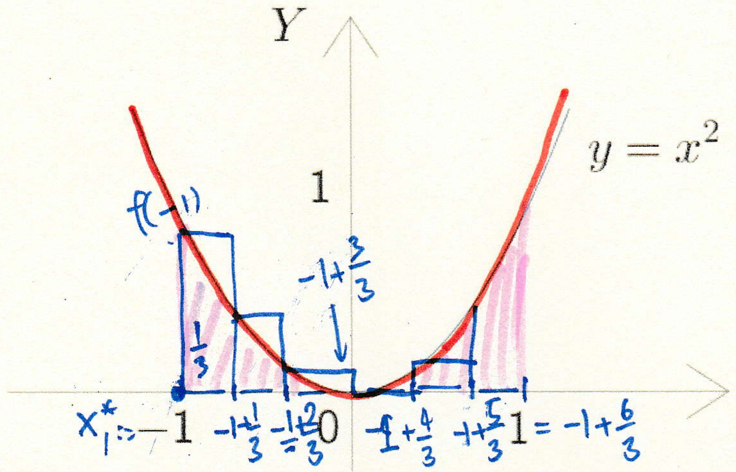


พ.ท. ▭ = กว้าง x สูง
$= \Delta x \cdot f(x_k^*)$

$$A_n = \Delta x \cdot f(x_1^*) + \Delta x \cdot f(x_2^*) + \dots + \Delta x \cdot f(x_n^*)$$
$$= (f(x_1^*) + f(x_2^*) + \dots + f(x_n^*))\Delta x$$
$$= \sum_{k=1}^{n} f(x_k^*)\Delta x$$
$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

(จุด x_k^* เลือกเป็นใครก็ได้ในช่วงย่อย)

It is probably easiest to see how we do this with an example. So let's determine the area between $f(x) = x^2$ on $[-1, 1]$. In other words, we want to determine the area of the shaded region below.



Δx ① แบ่ง $[-1, 1]$ ออกเป็น 6 ช่วงย่อย : ความกว้างช่วง $\Delta x = \frac{1-(-1)}{6} = \frac{2}{6} = \frac{1}{3}$

x_k^* ② เลือกความสูงของ ⊡ เป็นจุดปลายช่วงย่อยด้านซ้าย

$$A_6 = \left(\tfrac{1}{3}\right)f(-1) + \left(\tfrac{1}{3}\right)f\left(-1+\tfrac{1}{3}\right) + \left(\tfrac{1}{3}\right)f\left(-1+\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)f\left(-1+\tfrac{3}{3}\right) + \left(\tfrac{1}{3}\right)f\left(-1+\tfrac{4}{3}\right) + \left(\tfrac{1}{3}\right)f\left(-1+\tfrac{5}{3}\right)$$

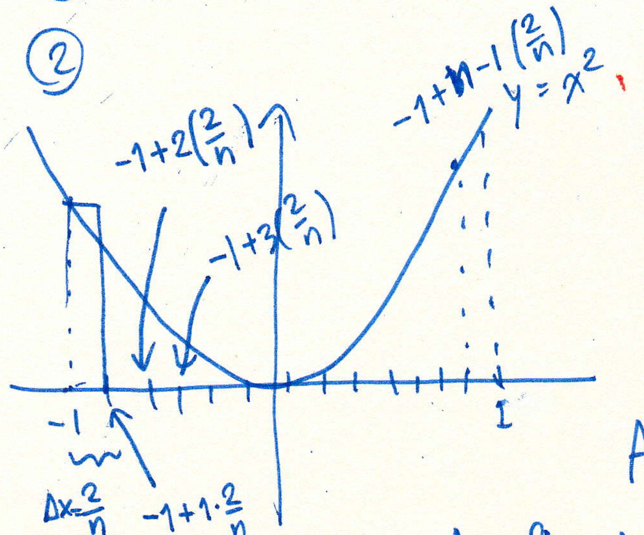
$$= f\left(-1+\tfrac{1}{3}(0)\right)\cdot\tfrac{1}{3} + f\left(-1+\tfrac{1}{3}(1)\right)\tfrac{1}{3} + f\left(-1+\tfrac{1}{3}(2)\right)\tfrac{1}{3} + \dots + f\left(-1+\tfrac{1}{3}(5)\right)\tfrac{1}{3}$$

$$A_6 = \sum_{k=0}^{5} f\left(-1+\tfrac{1}{3}k\right)\tfrac{1}{3}$$

แบ่ง n ช่วง

① แบ่ง $[-1, 1]$ ออกเป็น n ช่วง : ความกว้างช่วง คือ $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$

②



$$A_n = f(-1)\left(\tfrac{2}{n}\right) + f\left(-1+1\cdot\tfrac{2}{n}\right)\tfrac{2}{n} + \dots + f\left(-1+(n-1)\tfrac{2}{n}\right)\cdot\tfrac{2}{n}$$

$$= \sum_{k=0}^{n-1} \underbrace{f\left(-1+k\left(\tfrac{2}{n}\right)\right)}_{\text{ความสูง}} \underbrace{\left(\tfrac{2}{n}\right)}_{\text{ความกว้างช่วง}}$$

$$A_n = \sum_{k=0}^{n-1} \left(-1+k\left(\tfrac{2}{n}\right)\right)^2 \left(\tfrac{2}{n}\right) \qquad (f(x) = x^2)$$

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(-1+k\left(\tfrac{2}{n}\right)\right)^2\left(\tfrac{2}{n}\right) = \frac{2}{3}$$

<u>ปัญหาท้าทายปัญญา</u> $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(-1 + k\left(\frac{2}{n}\right)\right)^2 \left(\frac{2}{n}\right) = \frac{2}{3}$

Example 5.1

$\sum_{k=4}^{8} k^3 =$

$\sum_{k=0}^{5} (-1)^k (2k-1) =$

Ex $\sum_{k=1}^{10} 1 = \underbrace{1+1+\dots+1}_{10 \text{ ตัว}}$
$= 10 \cdot 1$

5.2.2 Properties of Sums

Theorem 5.1

(a) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

(b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

$\sum_{k=1}^{n} a = na$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$(1+2+3+4+\dots+n)$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$(1^2+2^2+3^2+\dots+n^2)$

Table 5.1 below shows the result of evaluating (5.1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to ....

n	6	10	100	1,000	10,000
A_n	0.7	0.68	0.6668	0.666668	0.66666668

Table 5.1: estimation of area

So, increasing the number of rectangles improves the accuracy of the estimation as we would guess. Later in this chapter we will show that

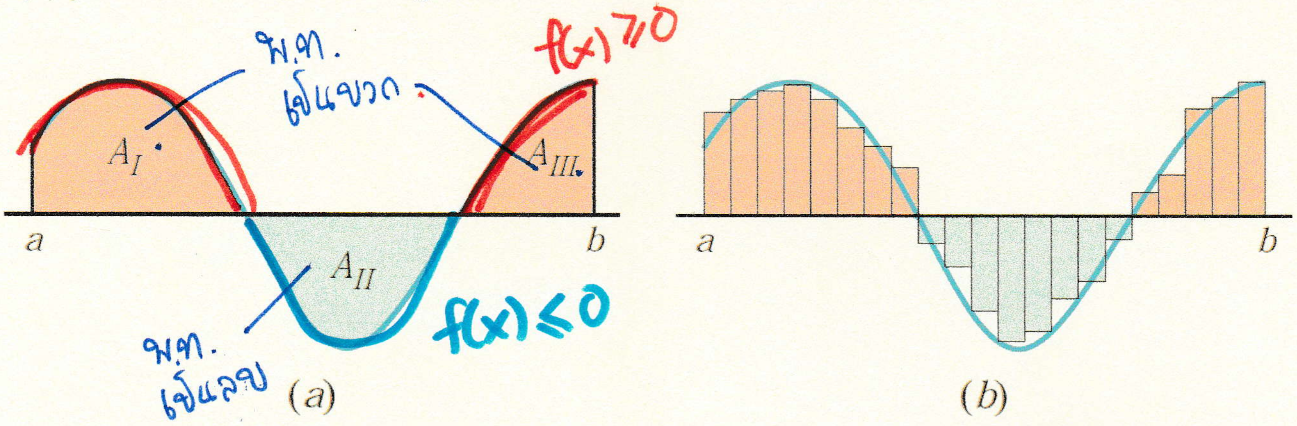
$$\lim_{n\to\infty} A_n = \frac{2}{3}.$$

5.2.5 $\boxed{\text{Net Signed Area}}$ พื้นที่ใต้กราฟ f(x) เป็นได้ทั้งบวก และลบ

If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x}$$

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas — the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the **net signed area**.



(a) (b)

For example, in Figure 5.5, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(AI + AIII) - AII = [\text{ area above } [a, b]] - [\text{ area below } [a, b]]$$

DEFINITION 5.2 (Net Signed Area) If the function f is continuous on $[a, b]$, then the net signed area A between $y = f(x)$ and the interval $[a, b]$ is defined by

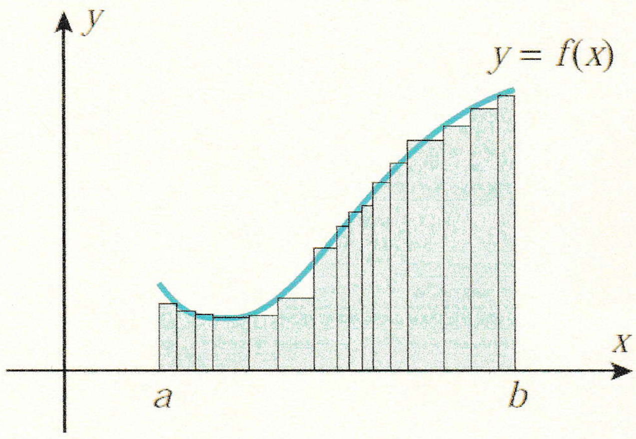
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

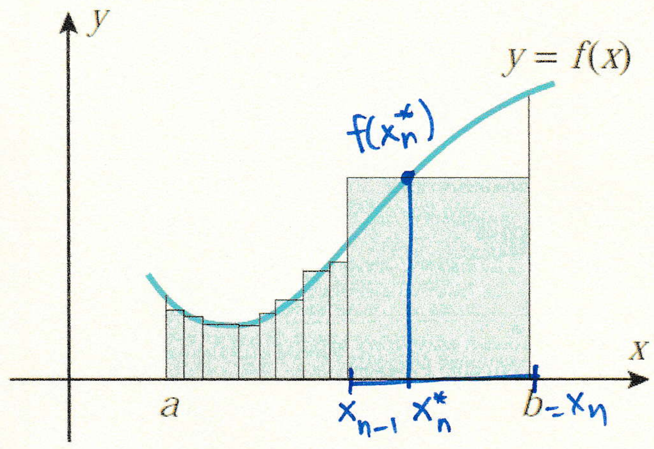
① พื้นที่ $(A_I + A_{III}) - (A_{II})$

② พื้นที่แบบ Net-signed area.

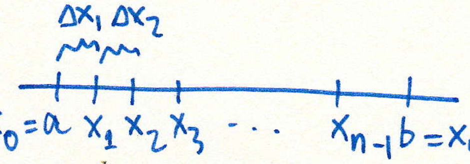
Definite Integral

: คือ พ.ท. ของ Net signed area.





A partition of the interval $[a, b]$ is a collection of points



$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

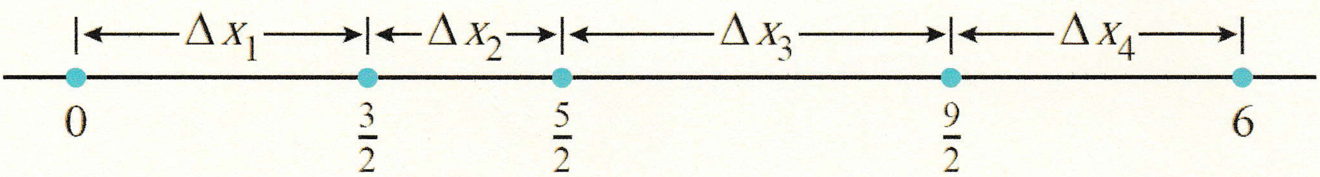
พ.ท. ▯ $= \Delta x_n \cdot f(x_n^*)$

that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1,\ \Delta x_3 = x_3 - x_2,\ \ldots,\ \Delta x_n = x_n - x_{n-1}$$

The partition is said to be regular provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}.$$



$$\boxed{\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2}$$

If we are to generalize Definition 5.2.4 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \text{ is replaced by } \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

We also need to replace the expression $n\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose.

พ.ท. $A_n = \Delta x_1 \cdot f(x_1^*) + \Delta x_2 \cdot f(x_2^*) + \dots + \Delta x_n F(x_n^*)$

$= \sum_{k=1}^{n} f(x_k^*)\Delta x_k \leftarrow$ ผลบวกของแท่ง n ชิ้น

$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$

DEFINITION A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

upper limit

ความกว้างช่วง

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

lower limit

integrand

function

which is called the *definite integral* of f from a to b. The numbers a and b are called the *lower limit of integration* and the *upper limit of integration* , respectively, and $f(x)$ is called the *integrand* .

สรุป $\int_a^b f(x)\,dx$ คือ $\underbrace{\text{พ.ท.ใต้กราฟ}}_{\text{Net-signed area}}$ ของ $f(x)$ จาก a ถึง b

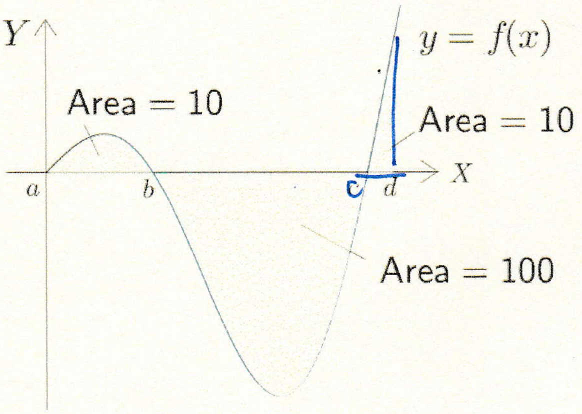
Theorem 5.2 If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x)dx.$$

Example 5.2 Use the areas shown in the figure to find

(a) $\int_a^b f(x)dx$ (b) $\int_b^c f(x)dx$ (c) $\int_a^c f(x)dx$ (d) $\int_a^d f(x)dx$

Solution



① $\int_a^b f(x)dx = 10$

② $\int_b^c f(x)dx = -100$

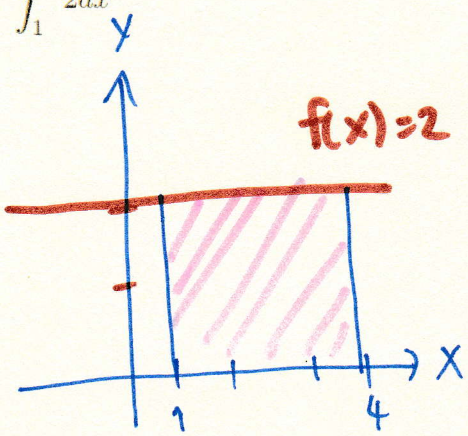
③ $\int_a^c f(x)dx = 10 + (-100)$
$= -90$

④ $\int_a^d f(x)dx = -80$

Example 5.3 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

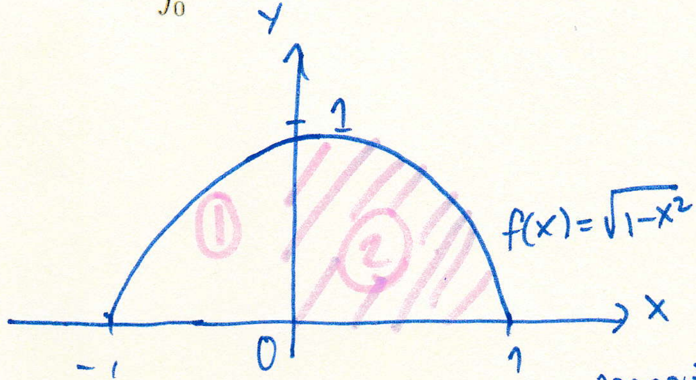
$f(x) = \sqrt{1-x^2}$

(a) $\displaystyle\int_1^4 2dx$



$$\int_1^4 2\,dx = (4-1)\times 2$$
$$= 3\times 2 = 6 \text{ ตร.นิ้ว}$$

(b) $\displaystyle\int_0^1 \sqrt{1-x^2}dx$



$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

วงกลมรัศมี 1 หน่วย

พ.ท. วงกลม

$= \pi(1)^2 = \pi$

ให้ $y = \sqrt{1-x^2}$; $y \geq 0$

$y^2 = 1-x^2$

$x^2+y^2 = 1$