Ex $\quad \frac{1}{2}\int 2\, 2^{\overset{u}{\sin(\arctan 2x)}} \frac{\cos(\arctan 2x)}{1+4x^2}\, dx$ du

$u = \sin(\arctan 2x)$; $du = \left[\cos(\arctan 2x)\frac{d}{dx}(\arctan 2x)\right]dx$

$\cancel{du = \cos(\arctan 2x)\frac{1}{\sqrt{1-(2x)^2}}\cdot 2}$

$du = \left(\cos(\arctan 2x)\cdot\frac{1}{1+(2x)^2}\cdot 2\right)dx$

$dx = \frac{(1+4x^2)}{2\cos(\arctan 2x)}du$

$\therefore \int 2^{\sin(\arctan 2x)}\frac{\cos(\arctan 2x)}{1+4x^2}dx = \int 2^u \frac{\cancel{\cos(\arctan 2x)}}{\cancel{1+4x^2}}\cdot\frac{\cancel{(1+4x)^2}}{2\cancel{\cos(\arctan 2x)}}du$

$= \frac{1}{2}\int 2^u du$

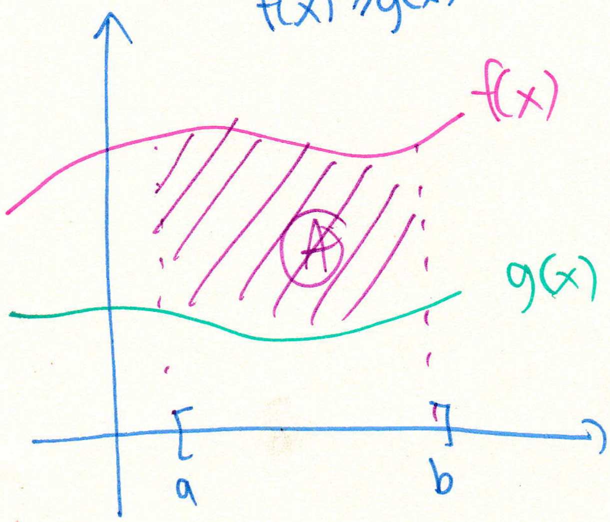
$= \frac{1}{2}\frac{2^u}{\ln 2} + C$

$= \frac{1}{2\ln 2}2^{\sin(\arctan 2x)} + C$ #

---

① Area between curves.

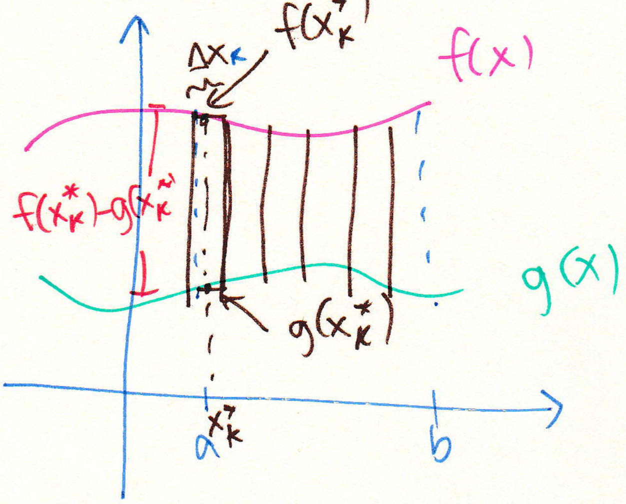
$f(x) \geq g(x)$



$A = \int_a^b [f(x) - g(x)]dx.$

② ปริมาตรของทรงตัน.

พ.ท.



พ.ท. 1 ชิ้นของเราคือ

= กว้าง x ยาว

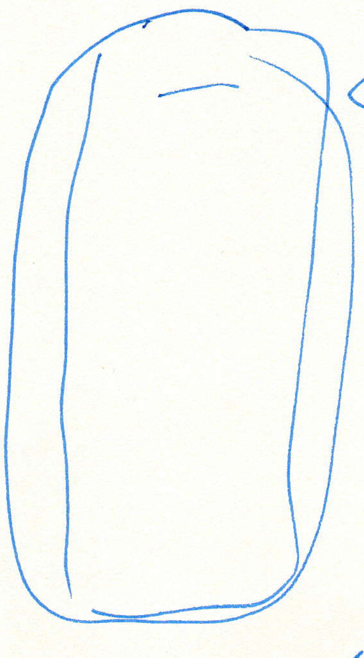
$$= \underbrace{[f(x_k^*)-g(x_k^*)]\Delta x_k}_{\text{หา } n \text{ ชิ้น}}$$

$$\text{พ.ท.} = \sum_{k=1}^{n} [f(x_k^*)-g(x_k^*)]\Delta x_k$$

$$\text{พ.ท.จริง} = \lim_{n\to\infty} \sum_{k=1}^{n} [f(x_k^*)-g(x_k^*)]\Delta x_k$$

$$= \int_a^b [f(x)-g(x)]dx.$$

ปริมาตร



พ.ท.
หน้าตัด
$A(x)$

ปริมาตรของ
ชิ้น 1 ชิ้นย่อย
คือ.

$A(x_k^*) . \Delta x_k$

∴ มี n ชิ้นที่ตัดแล้ว

หนา
Δx

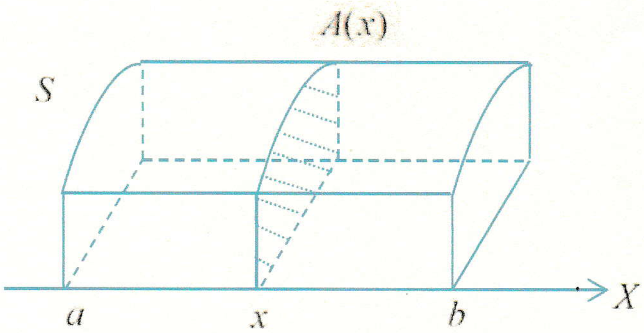
$$\therefore V_n = \sum_{k=1}^{n} \underbrace{A(x_k^*)}_{\text{พ.ท.หน้าตัด}} \underbrace{\Delta x_k}_{\text{ความหนา}}$$

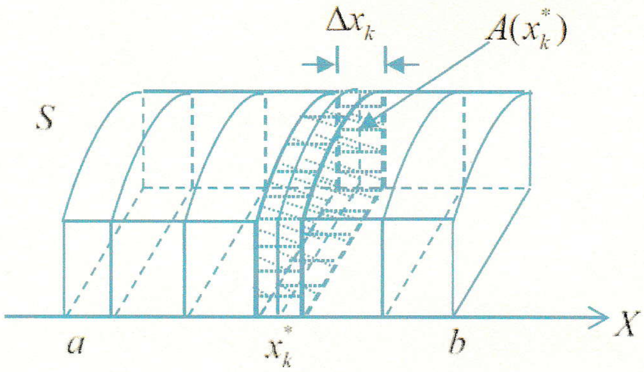
$$V = \int_a^b A(x)dx \longleftarrow \therefore V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x_k.$$

6.2 Volumes by slicing; Disks and Washers

Theorem 6.3 (**Volume formula**) Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in [a, b], the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)dx. \tag{6.3}$$





ปริมาตรชิ้นย่อย 1 ชิ้น

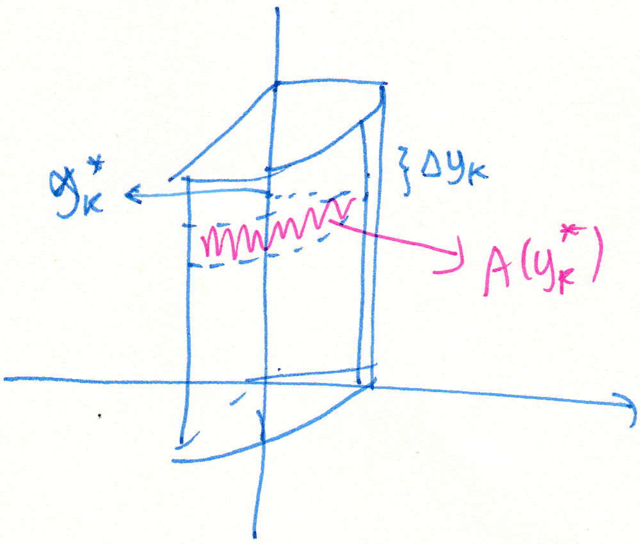
$$V_K = \Delta x_K \cdot A(x_K^*)$$

$$V = \lim_{max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)dx$$

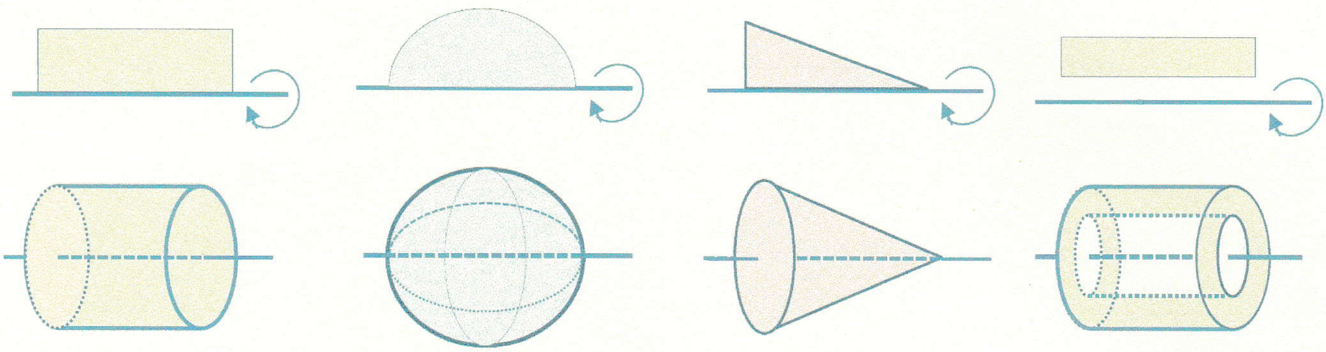
There is a similar result for cross sections perpendicular to the y-axis.

Theorem 6.4 (**Volume formula**) Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in [c, d], the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)dy. \tag{6.4}$$



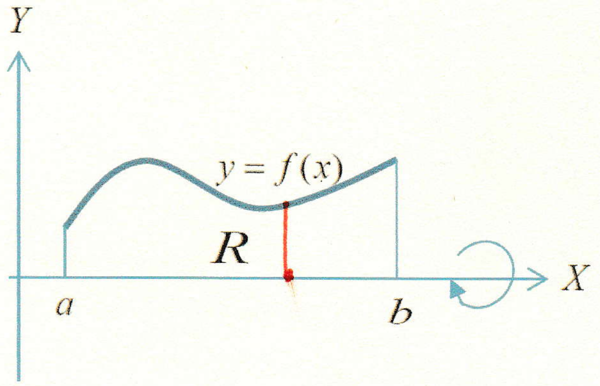
Solid of revolution

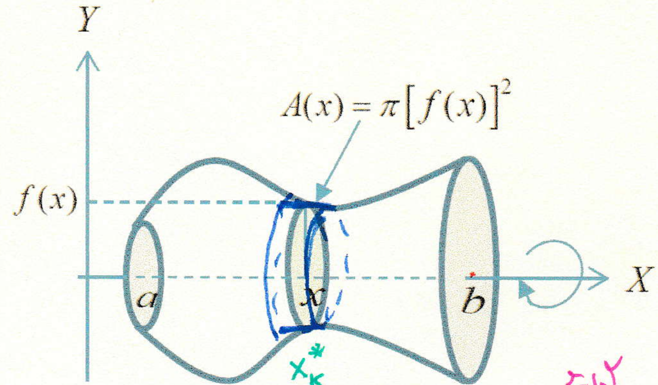


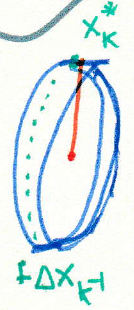
Volume by Disks perpendicular to the X-axis

$\boxed{\text{Disk method}}$ การหมุนแบบจาน

Problem: Let f be continuous (ต่อเนื่อง) and nonnegative (ไม่เป็นค่าลบ) on [a, b], and let R be the region that is bounded above (ปิดล้อมด้านบน) by $y = f(x)$, below (ด้านล่าง) by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution (ทรงตันจากการหมุน) that is generated by revolving the region R about the X-axis.







รัศมี

$A(x) = \pi [f(x)]^2$

ปริมาตรที่ x_k^* 1 ชิ้น

$V(x_k^*) = \underbrace{\pi [f(x_k^*)]^2}_{\text{พ.ท.หน้าตัด}} \underbrace{\Delta x_k}_{\text{ความหนา}}$

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is a circular disk of radius $f(x)$. The area of this region is

$$A(x) = \pi[f(x)]^2.$$

ปริมาตรทั้งก้อน

$V_n = \sum_{k=1}^{n} \pi [f(x_k^*)]^2 \Delta x_k$

Thus, from (6.3) the volume of the solid is

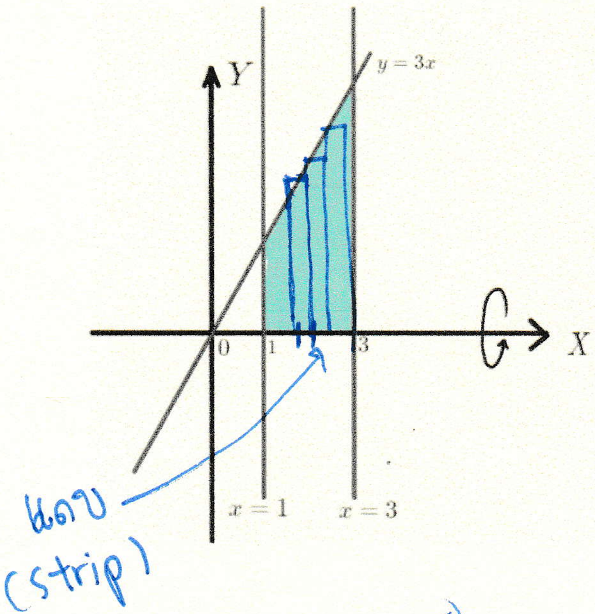
$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{รัศมี}} \underbrace{dx}_{\text{ความหนา}}. \tag{6.5}$$

$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \pi [f(x_k^*)]^2 \Delta x_k$

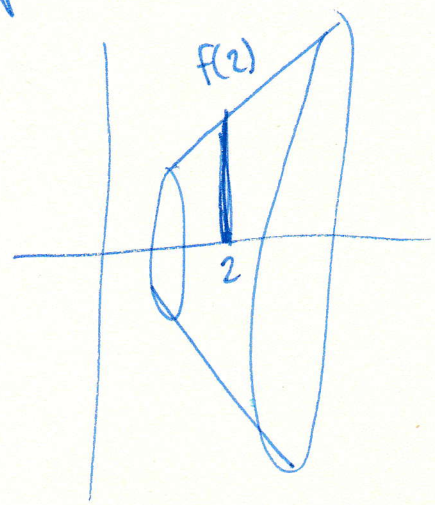
$V = \int \pi [f(x)]^2 dx$

Because the cross sections are disk shaped, the application of this formula is called the *method of disks*.

Example 6.6 Find the volume of the solid that is obtained when the region under the curve $y = 3x$ over the interval $[1, 3]$ is revolved about the X-axis.

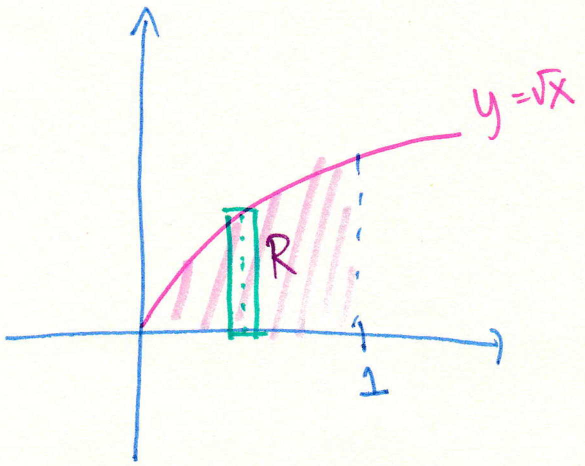


$$V = \int_1^3 \pi [3x]^2 \, dx$$
$$= \pi \int_1^3 9x^2 \, dx$$
$$= \pi \left[\frac{9x^3}{3} \right]_1^3$$
$$= \pi [3 \cdot 3^3 - 3 \cdot 1^3]$$
$$= \pi [81 - 3]$$
$$= \pi [78] = 78\pi. \quad \#$$



(*) การหาปริมาตรแบบ Disk Method จะทำการสร้างแถบ (strip) รูปสี่เหลี่ยมผืนผ้า จะสังเกตว่า การสร้างแถบ จะสร้างแถบให้ตั้งฉาก กับแกนหมุน

Ex ให้ $f(x) = \sqrt{x}$ จาก $x = 0$ ถึง $x = 1$, หาปริมาตรของบริเวณที่ปิดล้อมด้วย $f(x)$ จาก $x = 0$ ถึง $x = 1$



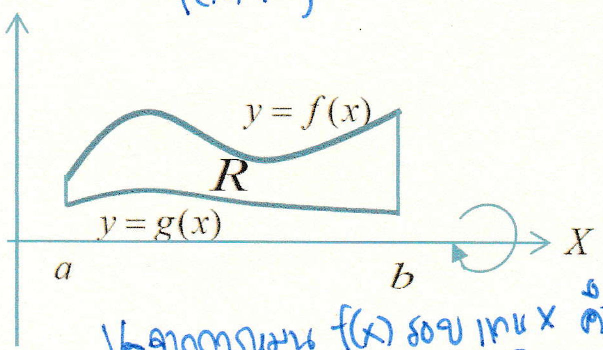
พื้นที่หน้าตัด

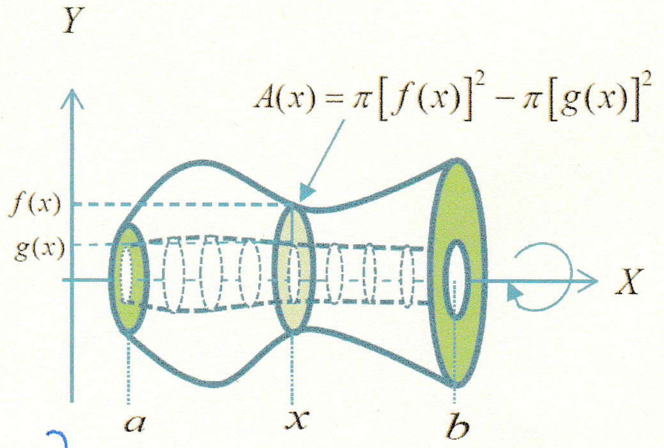
$$V = \int_0^1 \pi (\sqrt{x})^2 \, dx$$
$$= \pi \int_0^1 x \, dx$$
$$= \pi \left[\frac{x^2}{2} \right]_0^1$$
$$= \pi \left[\frac{1}{2} - \frac{0}{2} \right] = \frac{\pi}{2} \quad \#$$

Volume by Washers perpendicular to the X-axis

Problem: Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the X-axis.

$f(x) \geq g(x)$





V_f จากการหมุน $f(x)$ รอบแกน x คือ

$$V_f = \int_a^b \pi [f(x)]^2 \, dx$$

V_g จากการหมุน $g(x)$ รอบแกน x คือ

$$V_g = \int_a^b \pi [g(x)]^2 \, dx$$

V ที่ได้ $= V_f - V_g$

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is the annular or "washer-shaped", region with inner radius $g(x)$ and outer radius $f(x)$. The area of this region is

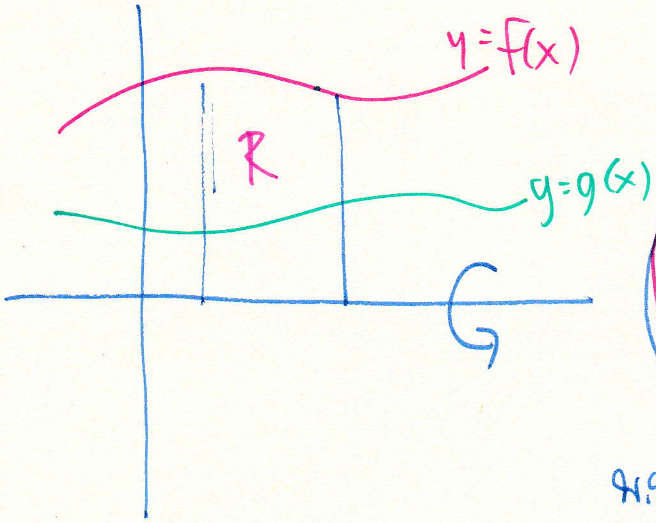
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi\left([f(x)]^2 - [g(x)]^2\right)$$

$$V = \int_a^b \pi [f(x)]^2 \, dx - \int_a^b \pi [g(x)]^2 \, dx = \int_a^b \pi \left[(f(x))^2 - g(x)^2 \right] dx.$$

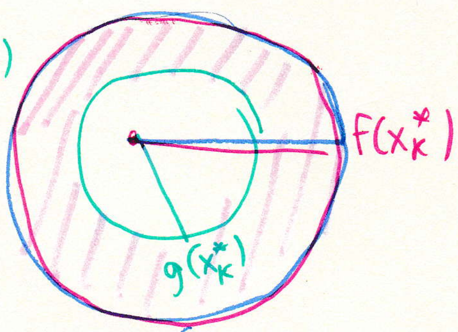
Thus, from (6.3) the volume of the solid is

$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx \tag{6.6}$$

Because the cross sections are washer shaped, the application of this formula is called the *method of washers.*



พ.ท. หน้าตัดที่ x_k^*



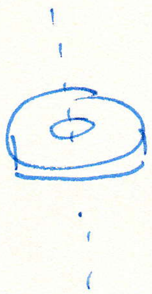
พ.ท. หน้าตัด เป็นวงแหวน.

มี พ.ท. คือ

$$\pi[f(x_k^*)]^2 - \pi[g(x_k^*)]^2$$

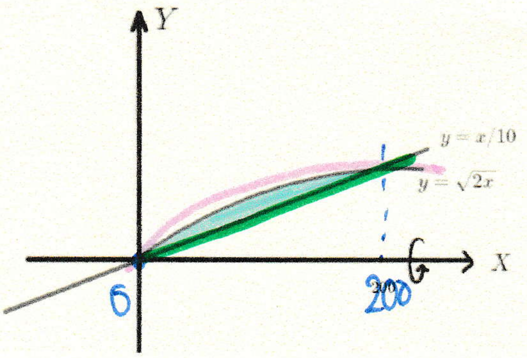
พ.ท. หน้าตัด × ความสูงแต่ละแผ่น dx

$$\therefore V = \int_a^b \left[\pi[f(x_k^*)]^2 - \pi[g(x_k^*)]^2\right] dx$$



Washer method / ring method

Example 6.7 Find the volume of the solid that is obtained when the region between the graphs of the equations $y = \sqrt{2x}$ and $y = \frac{x}{10}$ over the interval $[0, 200]$ is revolved about the X-axis.



จุดตัดของกราฟ

$y = \sqrt{2x}$ กับ $y = \frac{x}{10}$

Set $\sqrt{2x} = \frac{x}{10}$

$$2x = \frac{x^2}{100}$$

$$200x - x^2 = 0$$

$$x^2 - 200x = 0$$

$$x(x-200) = 0 \Rightarrow x = 0,\ x = 200$$

$$V = \int_0^{200} \pi \left[\left(\sqrt{2x}\right)^2 - \left(\frac{x}{10}\right)^2 \right] dx$$

$$= \pi \int_0^{200} \left[2x - \frac{x^2}{100} \right] dx$$

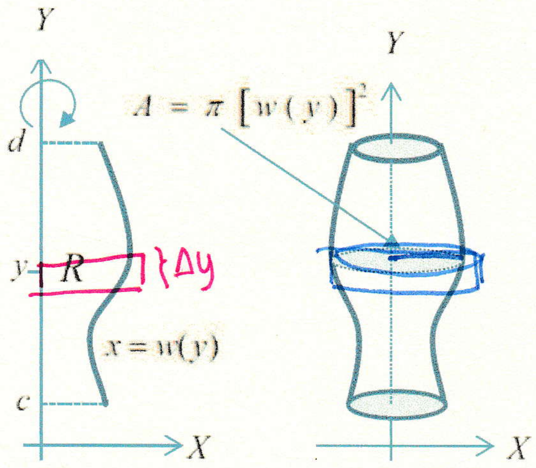
$$= \pi \left[x^2 - \frac{x^3}{300} \right]_0^{200}$$

$$\vdots$$

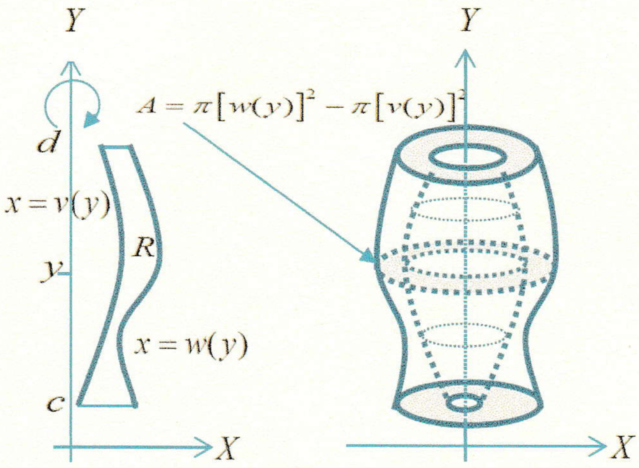
$$= \pi \left[\left[(200)^2 - \frac{(200)^3}{300} \right] - \left[0^2 - \frac{0^3}{300} \right] \right]$$

$$= \pi \left[(200)^2 - \frac{(200)^3}{300} \right] = \frac{40{,}000\pi}{3}. \quad \#$$

Volume by Disks and Washers perpendicular to the Y-axis



Disks



Washers

The methods of disks and washers have analogs for regions that are revolved about the Y-axis. Using the method of slicing and Formula (6.4), the following formulas for the volumes of the solid are

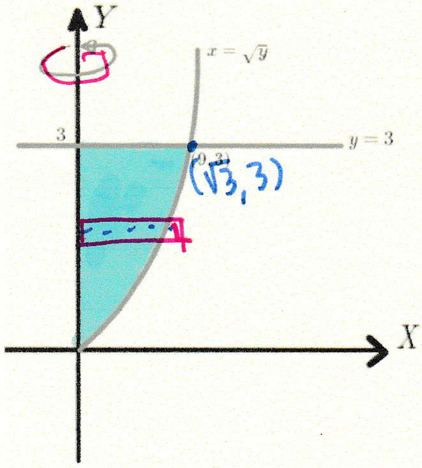
$$V = \int_c^d \pi [w(y)]^2 dy \qquad (disks), \tag{6.7}$$

$$V = \int_c^d \pi \left([w(y)]^2 - [v(y)]^2 \right) dy \qquad (washers). \tag{6.8}$$

ข้อควรระวัง ถ้าหมุนรอบแกน y ต้องจัดรูปแบบ $x = w(y)$

ขอบเขตการอินทิเกรต : ดูที่ขอบเขตของ y.

Example 6.8 Find the volume of the solid generated when the region enclosed by $x = \sqrt{y}$, $y = 0$, and $y = 3$ is revolved about the Y-axis.



จุดตัดของ $x = \sqrt{y}$ → ถ้า $y = 3$ ได้ $x = \sqrt{3}$

(รอบแกน y)

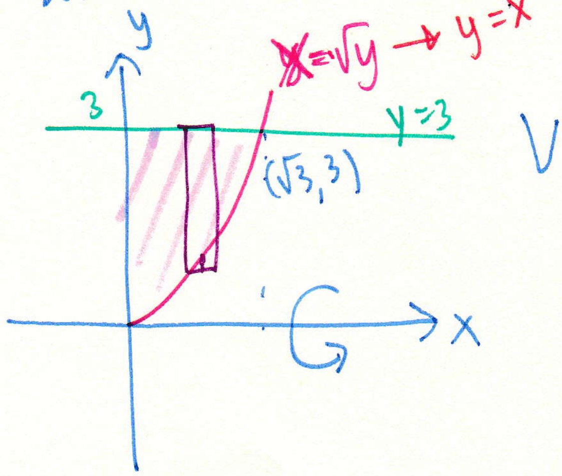
$$V = \int_0^3 \pi(\sqrt{y})^2\,dy$$

$$= \pi\int_0^3 y\,dy$$

$$= \pi\left[\frac{y^2}{2}\Big|_0^3\right]$$

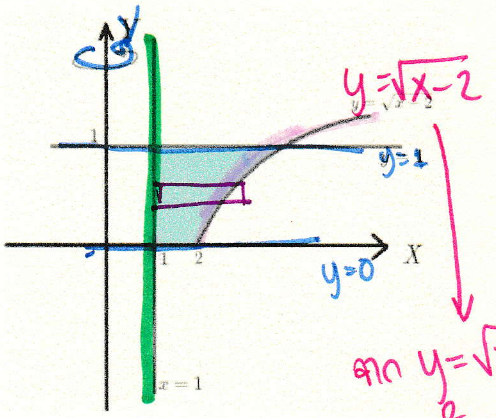
$$= \pi\left(\frac{9}{2} - \frac{0}{2}\right) = \frac{9}{2}\pi \quad \#$$

หาปริมาตรหมุนรอบแกน X ด้วย Washer



$$V = \int_0^{\sqrt{3}} \pi\left[(3)^2 - (x^2)^2\right]dx$$

Example 6.9 Find the volume of the solid generated when the region enclosed by $x = 1$, $y = \sqrt{x-2}$, $y = 0$, and $y = 1$ is revolved about the Y-axis.



$$\boxed{V = \int_0^1 \pi\left[(y^2+2)^2 - (1)^2\right] dy}$$

จาก $y = \sqrt{x-2}$

$y^2 = x - 2$

$\therefore x = y^2 + 2$

$$V = \pi \int_0^1 \left[y^4 + 4y^2 + 4 - 1\right] dy$$

$$= \pi \int_0^1 \left[y^4 + 4y^2 + 3\right] dy$$

$\vdots$

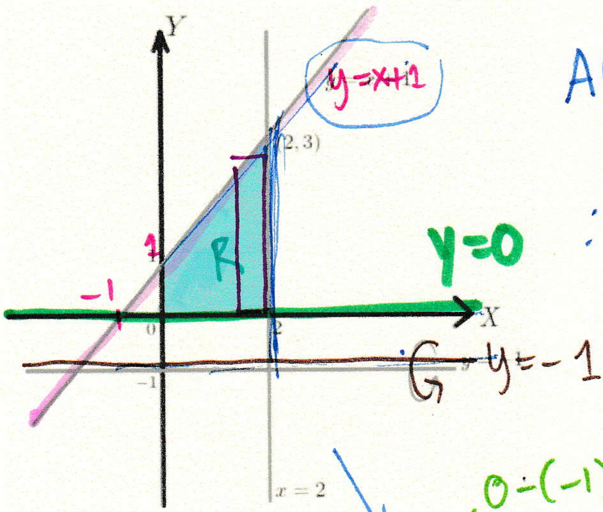
$$V = \frac{58}{15}\pi \ .$$

#

Other axes of revolution

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (6.3) and (6.4) and integrate an appropriate cross-sectional area to find the volume.

Example 6.10 Find the volume of the solid that is obtained when the region between the curve $y = x + 1$ and $y = 0$ over the interval $[0, 2]$ is rotated about the line $y = -1$.



$$A(x) = \pi\{[(x+1)-(-1)]^2 - [0-(-1)]^2\}$$

$$\therefore V = \pi\int_0^2 [[(x+1)-(-1)]^2 - [0-(-1)]^2]\,dx$$

$$V = \pi\int_0^2 [(x+2)^2 - (1)^2]\,dx$$

$$\vdots$$

$$V = \frac{50\pi}{3} \#$$