1 Integration by parts

- $\int x \cos x \, dx$
- $\int x e^{x^2} dx \longrightarrow u = x^2 \; ; \; du = 2x\,dx$

  $\vdots$

$\downarrow$

$\int x e^{x} dx$

$$d(uv) = u\,dv + v\,du$$

$$\int d(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\therefore \boxed{\int u\,dv = uv - \int v\,du}$$

หาว่าใครเป็น u, ใครเป็น dv

$u \downarrow$ diff ง่ายๆ

$dv \downarrow$ อินทิเกรตง่ายๆ

Ex $\int x \sin x \, dx$

$u = x \quad ; \; dv = \sin x \, dx$

$du = dx \quad ; \; v = -\cos x$

หา du, v

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\int x \sin x \, dx = -x\cos x - \int (-\cos x)\, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C \quad \#$$

Ex $\int \ln x \, dx$ ใช้ by parts

$$u = \ln x \quad ; \quad dv = dx$$
$$du = \frac{1}{x}dx \quad ; \quad v = x$$

$$\therefore \int \ln x \, dx = x\ln x - \int x \cdot \frac{1}{x} dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C \qquad \#$$

Algebraic function

$\boxed{\text{LIATE}}$ → exponential function.

trigonometric function

inverse trigone

Logarithmic function

Ex $\int x^3 \ln x \, dx$

$$u = \ln x \quad ; \quad dv = x^3 dx$$
$$du = \frac{1}{x}dx \quad ; \quad v = \frac{x^4}{4}$$

$$\therefore \int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} dx$$
$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$$
$$= \frac{x^4}{4}\ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + C. \qquad \#$$

7.2.2 Repeated integration by parts

Example 7.3 Evaluate $\int x^2e^x\,dx$.

$u = e^x \qquad ; \quad dv = x^2\,dx$

$du = e^x\,dx \qquad ; \quad v = \dfrac{x^3}{3}$

$$\therefore \int x^2e^x\,dx = e^x\cdot\frac{x^3}{3} - \int \frac{x^3}{3}e^x\,dx$$

$u = x^2 \qquad ; \quad dv = e^x\,dx$

$du = 2x\,dx \qquad ; \quad v = e^x$

$$\therefore \int x^2e^x\,dx = e^x\cdot x^2 - \boxed{\int 2xe^x\,dx}$$

$$= e^x\cdot x^2 - 2\boxed{\int xe^x\,dx} \rightarrow (*)$$

พิจารณา $\int xe^x\,dx$

$u_1 = x \qquad ; \quad dv_1 = e^x\,dx$

$du_1 = dx \qquad ; \quad v_1 = e^x$

$$\therefore \int xe^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C_1$$

$$\therefore \int x^2e^x\,dx = e^x\cdot x^2 - 2(xe^x - e^x + C_1)$$

$$= e^x\cdot x^2 - 2xe^x - 2e^x + C \quad \#$$

<u>ข้อสังเกต</u> ในการอินทิเกรต by parts ซ้ำหลายๆรอบ
ควรเลือก u, dv เป็นฟังก์ชันประเภทเดิม.

ปัญหาท้าทายปัญญา Reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Hint แยก $\cos^n x = \cos^{n-1}x \cdot \cos x$

ใช้ integration by parts

$u = \cos^{n-1} x$

$dv = \cos x\,dx$

ใช้เอกลักษณ์ $\sin^2 x = 1-\cos^2 x$

LIATE

Example 7.4 Evaluate $\int e^x \cos x \, dx$.

$$u = \cos x \quad ; \quad dv = e^x dx$$
$$du = -\sin x \, dx \quad ; \quad v = e^x$$

$$\therefore \int e^x \cos x \, dx = e^x \cos x - \int e^x(-\sin x) dx$$
$$= e^x \cos x + \boxed{\int e^x \sin x \, dx}$$

<u>by parts</u> พิจารณา $\int e^x \sin x \, dx$

$$u_1 = \sin x \quad ; \quad dv_1 = e^x dx$$
$$du_1 = \cos x \, dx \quad ; \quad v_1 = e^x$$

$$\therefore \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

แทนค่ากลับเดิม

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx.$$
$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$
$$\therefore \int e^x \cos x \, dx = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C \qquad \#$$

7.2.3 Integration by parts for definite integrals

$\int \tan^{-1} x \, dx$

Example 7.5 Evaluate $\int_0^1 \tan^{-1} x \, dx$.

$u = \arctan x \quad ; \quad dv = dx$

$du = \frac{1}{1+x^2} dx \quad ; \quad v = x$

$$\therefore \int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \boxed{\int_0^1 \frac{x}{1+x^2} dx} \longrightarrow (*)$$

ทด

อินทิเกรตโดยเปลี่ยนตัวแปร

พิจารณา $\frac{1}{2}\int_0^1 \frac{2x}{1+x^2} dx$

$\heartsuit = 1+x^2 \quad ; \quad d\heartsuit = 2x\,dx$

$x=0 \longrightarrow \heartsuit = 1$

$x=1 \longrightarrow \heartsuit = 2$

$$\frac{1}{2}\int_0^1 \frac{2x}{1+x^2} dx = \frac{1}{2}\int_1^2 \frac{1}{\heartsuit} d\heartsuit$$

$$= \frac{1}{2}\Big[\ln|\heartsuit|\Big]_1^2$$

$$= x \arctan x \Big|_0^1 - \frac{1}{2}\ln|\heartsuit|\Big|_1^2$$

$$= \Big[1 \cdot \arctan 1 - 0 \cdot \cancel{\arctan 0}\Big] - \frac{1}{2}\Big[\ln|2| - \cancel{\ln|1|}\Big]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln|2| \quad //$$

$$\left(= \frac{\pi}{4} - \ln\sqrt{2} \right)$$

$\sin^2 x + \cos^2 x = 1$

หารด้วย $\sin^2 x$ ทั้งสมการ

$1 + \cot^2 x = \mathrm{cosec}^2 x$

หารด้วย $\cos^2$ ทั้งสมการ

$\tan^2 x + 1 = \sec^2 x$

7.3 Integrating trigonometric functions

We start the section by reviewing important trigonometric identities as following:

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x = \sec^2 x - 1$
$\sin 2x = 2 \sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$\sin A \pm B = \sin A \cos B \pm \cos A \sin B$	$\cos A \pm B = \cos A \cos B \mp \sin A \sin B$

7.3.1 Integrating products of sines and cosines

If m and n are positive integers, the integral $\boxed{\int \sin^m x \cos^n x \, dx}$ can be evaluated by one of the following procedures, depending on whether m and n are even or odd.

$\int \sin^m x \cos^n x \, dx$	Procedure	relevant identity
m odd	set $\sin^m x = \sin^{m-1} x \sin x$	$\sin^2 x = 1 - \cos^2 x$
n odd	set $\cos^n x = \cos^{n-1} x \cos x$	$\cos^2 x = 1 - \sin^2 x$
m and n even	set $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or set $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example 7.6 Evaluate $\int \sin^3 x \, dx$.

$\boxed{\sin^2 x + \cos^2 x = 1}$

$\sin^2 x = 1 - \cos^2 x$

$$\begin{aligned}
\int \sin^3 x \, dx &= \int (\sin x) \sin^2 x \, dx \\
&= -\int (-\sin x)[1 - \cos^2 x] dx
\end{aligned}$$

$u = \cos x$; $du = -\sin x \, dx$

$$\begin{aligned}
&= -\int [1 - u^2] du \\
&= \int (u^2 - 1) du \\
&= \frac{u^3}{3} - u + C \\
&= \frac{\cos^3 x}{3} - \cos x + C \quad \#
\end{aligned}$$

$u = mx$

$du = \frac{1}{m} dx$

$\int \cos mx \, dx = \frac{1}{m} \int \cos u \, du$

$= \frac{1}{m} \sin mx + C$

$\int \cos mx \, dx = \frac{\sin mx}{m} + C$

$\int \sin mx \, dx = \frac{-\cos mx}{m} + C.$

Ex

$\int \cos 6x \, dx$

$= \frac{\sin 6x}{6} + C.$

Example 7.7 Evaluate $\int \cos^5 x\, dx$.

$$\int \cos^5 x\, dx = \int \cos x \cdot \cos^4 x\, dx$$

$$= \int \cos x \left[\cos^2 x\right]^2 dx$$

$u = \sin x$

$du = \cos x\, dx$

$$= \int \cos x \left[1 - \overset{u}{\sin}^2 x\right]^2 dx$$

$$= \int (1-u^2)^2\, du$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C \quad \#$$

Example 7.8 Evaluate $\int \sin^2 x \cos^3 x\, dx$.

$\boxed{\cos^2 x = 1 - \sin^2 x}$

$$\int \sin^2 x \cos^3 x\, dx = \int (\sin^2 x)(\cos^2 x) \cos x\, dx$$

$$= \int (\sin^2 x)\left[1 - \sin^2 x\right] \cos x\, dx$$

$$= \int (\overset{u}{\sin}^2 x - \overset{u}{\sin}^4 x)\, \overset{du}{\cos x\, dx}$$

$u = \sin x$; $du = \cos x\, dx$

$$= \int (u^2 - u^4)\, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{(\sin^3 x)}{3} - \frac{\sin^5 x}{5} + C \quad \#$$

$$\boxed{\int \sin^m x \cdot \cos^n x\, dx}$$

Ex $\int \sin^2 x \cos^2 x \, dx$ $\quad$ $\sin^2 x = 1 - \cos^2 x$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1}{2}(1-\cos 2x) \\ \cos^2 x &= \frac{1}{2}(1+\cos 2x) \end{aligned}}$$

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int (1-\cos^2 x)\cos^2 x \, dx \\
&= \int (\cos^2 x - \cos^4 x) dx \\
&= \boxed{\int \cos^2 x \, dx} - \int \cos^4 x \, dx \\
&= \int \frac{1}{2}(1+\cos 2x) dx - \int (\cos^2 x)^2 dx \\
&= \frac{1}{2}\int (1+\cos 2x) dx - \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx \\
&= \frac{1}{2}x + \frac{1}{2}\frac{\sin 2x}{2} - \frac{1}{4}\int (1+\cos 2x)^2 dx \\
&= \frac{1}{2}x + \frac{\sin 2x}{4} - \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx \\
&= \frac{1}{2}x + \frac{\sin 2x}{4} - \frac{1}{4}x - \frac{\cancel{2}}{4} \cdot \frac{\sin 2x}{\cancel{2}} - \frac{1}{4}\int \frac{1}{2}(1+\cos 4x) dx \\
&= \frac{1}{2}x + \frac{\sin 2x}{4} - \frac{1}{4}x - \frac{\sin 2x}{4} - \frac{1}{8}\left[x + \frac{\sin 4x}{4}\right] + C \\
&= \left(\frac{1}{2}x - \frac{1}{4}x - \frac{1}{8}x\right) - \frac{1}{32}\sin 4x + C.
\end{aligned}$$