ทบทวน

Ex $\int x e^x \, dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$\int x e^x \, dx = x e^x - \int e^x \, dx$$

$$= x e^x - e^x + C$$

$\int x \arctan x \, dx \quad ; \quad u = x \qquad dv = \arctan x \, dx$

$du = dx \qquad v =$

$\left(\int \frac{1}{1+x^2} dx = \arctan x + C \right)$

$u = \arctan x \; ; \quad dv = x \, dx$

$du = \frac{1}{1+x^2} dx \; ; \quad v = \frac{x^2}{2}$

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$\int \frac{x^2}{1+x^2} dx \qquad \int \frac{1}{1+x^2} dx$

$$= \int \left(\frac{(1+x^2) - 1}{1+x^2} \right) dx$$

$$= \int \left(\cancel{\frac{1+x^2}{1+x^2}} \right) dx - \int \frac{1}{1+x^2} dx$$

$$= \int dx - \int \frac{1}{1+x^2} dx$$

$$= x - \arctan x + C.$$

$\int \ln(x^2+4) \, dx \; ; \quad u = \ln(x^2+4) \qquad dv = dx$

$du = \frac{2x}{x^2+4} dx \; ; \quad v = x$

HW?

$$\therefore \int \ln(x^2+4) \, dx = x \ln(x^2+4) - \boxed{\int \frac{2x^2}{x^2+4} dx}$$

Ex

$$\int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \cdot \sin^2 x \cdot \sin x \, dx \qquad \boxed{\sin^2 x = 1 - \cos^2 x}$$

$$= -\int \overset{u}{\cos^4 x} \left(1 - \overset{u^2}{\cos^2 x}\right) \overset{du}{(-\sin x \, dx)}$$

$$u = \cos x \quad ; \quad du = -\sin x \, dx$$

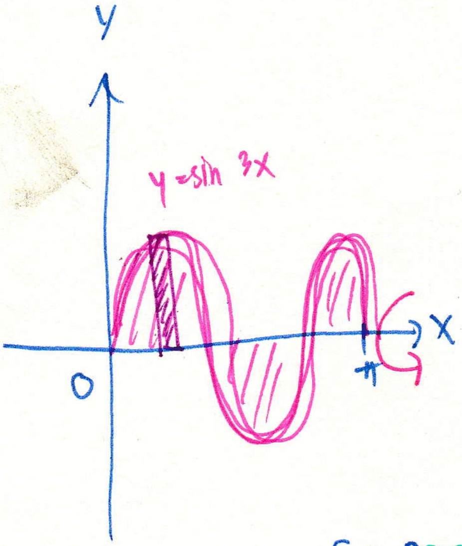
$$= -\int u^4 (1 - u^2) \, du$$

$$= -\int (u^4 - u^6) \, du$$

$$= -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C$$

$$= -\left(\frac{\cos^5 x}{5} - \frac{\cos^7 x}{7}\right) + C \qquad \#$$

Example 7.9 Find the volume V of the solid that is obtained when the region under the curve $y = \sin 3x$ over the interval $[0, \pi]$ is revolved about the x-axis.



$y = \sin x$ $\boxed{1 \text{ คาบ}} \rightarrow 2\pi$

$y = \sin 3x$

$$V = \int_0^\pi \pi [\sin 3x]^2 \, dx$$

$$= \pi \int_0^\pi \boxed{\sin^2 3x \, dx}$$

$\sin^2 ♡ = \frac{1}{2}(1 - \cos 2♡)$

พิจารณา $\pi \int \sin^2 3x \, dx = \pi \int \frac{1}{2}(1 - \cos 6x) \, dx$ (2♡)

$$= \frac{\pi}{2} \int (1 - \cos 6x) \, dx$$

$$= \frac{\pi}{2} \left[x - \frac{\sin 6x}{6} \right] + C$$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \end{aligned}}$$

$\int \cos mx \, dx = \frac{\sin mx}{m} + C$

$\int \sin mx \, dx = \frac{-\cos mx}{m} + C$

พิจารณา $\pi \int_0^\pi \sin^2 3x \, dx = \frac{\pi}{2} \left[x - \frac{\sin 6x}{6} \right]_0^\pi$

$$= \frac{\pi}{2} \left[\pi - \cancel{\frac{\sin 6\pi}{6}} - \cancel{0} + \cancel{\frac{\sin 0}{6}} \right]$$

$$= \frac{\pi^2}{2} \quad \#$$

Integrals of the form

$$\int \sin mx \cos nx \; dx, \quad \int \sin mx \sin nx \; dx, \quad \int \cos mx \cos nx \; dx$$

can be evaluated using the following identities:

$\sin(mx)\cos(nx) = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$	—1
$\sin(mx)\sin(nx) = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$	—2
$\cos(mx)\cos(nx) = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}$	—3

Example 7.10 Evaluate $\int \sin 3x \; \cos 5x \; dx$.

$$\int \sin(3x)\cos(5x)\,dx = \int \frac{1}{2}[\sin(3x+5x) + \sin(3x-5x)]\,dx$$

$$= \int \frac{1}{2}(\sin(8x) + \sin(-2x))\,dx$$

$$= \frac{1}{2}\left[\int \sin(8x)\,dx + \int \sin(-2x)\,dx\right]$$

$$\boxed{\begin{aligned}\sin(-x) &= -\sin x\\ \cos(-x) &= \cos x\end{aligned}}$$

$$= \frac{1}{2}\left[\int \sin(8x)\,dx - \int \sin(2x)\,dx\right]$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + C \quad \#$$

Ex $$\int \sin(7x)\sin(3x)\,dx = \int \frac{1}{2}[\cos(7x-3x) - \cos(7x+3x)]\,dx$$

$$= \frac{1}{2}\int(\cos(4x) - \cos(10x))\,dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 10x}{10}\right] + C \quad \#$$

Example 7.12 Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$.

① $\boxed{\text{Form 1}}$ มี $\sqrt{9-x^2}$

$$\sqrt{3^2 - x^2}$$

$a = 3$ $\quad u = x \longrightarrow u = a\sin\theta$

② $\boxed{\text{สมมติ}}$ $\boxed{x = 3\sin\theta}$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx$$

③ $\boxed{\text{แทนทุกอย่างที่เป็น } x \text{ ให้เป็นเทอมของ } \theta}$

ถ้า $x = 3\sin\theta \Rightarrow \boxed{dx = 3\cos\theta\, d\theta}$

$$\therefore \int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^3}{\sqrt{9-(3\sin\theta)^2}}\,3\cos\theta\,d\theta$$

④ $\boxed{\text{ใช้เอกลักษณ์ตรีโกณช่วย}}$

$$= 81\int \frac{\sin^3\theta\cdot\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= 81\int \frac{\sin^3\theta\cos\theta\,d\theta}{\sqrt{9(1-\sin^2\theta)}}$$

$$= \frac{\cancel{81}^{\,27}}{\cancel{3}_{\,1}}\int \frac{\sin^3\theta\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \longrightarrow 1-\sin^2\theta = \cos^2\theta$$

$$= 27\int \frac{\sin^3\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta$$

⑤ $\boxed{\text{อินทิเกรตให้ออก}}$

อินทิเกรตฟังก์ชันตรีโกณ $\longrightarrow$ $\displaystyle = 27\int \sin^3\theta\,d\theta$

$$= 27\int \sin^2\theta\,\underbrace{\sin\theta\,d\theta}_{du}$$

เปลี่ยนตัวแปร u $\searrow$ $\displaystyle = -27\int (1-\underbrace{\cos^2\theta}_{u^2})\,\sin\theta\,d\theta$ $\quad (u = \cos\theta\,;\ du = -\sin\theta\,d\theta)$

$$= -27\int (1-u^2)\,du$$

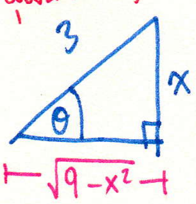
$$= -27\left[u - \frac{u^3}{3}\right] + C$$

แทนค่ากลับคืน $\longrightarrow$ $\displaystyle = -27\left[\cos\theta - \frac{\cos^3\theta}{3}\right] + C \longrightarrow (*)$

⑥ $\boxed{\text{แทนค่ากลับคืนในเทอม } x}$ (ใช้ Δ มุมฉากช่วย)

$x = 3\sin\theta \longrightarrow$

$\sin\theta = \dfrac{x}{3}$



$\therefore \cos\theta = \dfrac{\sqrt{9-x^2}}{3}$

$$(*) = -27\left[\frac{\sqrt{9-x^2}}{3} - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C$$

Ex $\displaystyle\int \frac{16}{x^2\sqrt{x^2-16}}\,dx$ (FORM 3)

$u = x \quad a = 4$

สมมติ $x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$

$$\int \frac{16}{x^2\sqrt{x^2-16}}\,dx = \int \frac{16}{(4\sec\theta)^2\sqrt{(4\sec\theta)^2-16}}\{4\sec\theta\tan\theta\,d\theta\}$$

$$= \int \frac{\cancel{16}\cdot\cancel{4}\sec\theta\tan\theta\,d\theta}{\cancel{16}\sec^2\theta\sqrt{\cancel{16}(\sec^2\theta-1)}}$$

$$= \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}$$

$\boxed{\sec^2\theta - 1 = \tan^2\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\tan^2\theta}}$$

$$= \int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\sec^{\cancel{2}}\theta\cdot\cancel{\tan\theta}}$$

$$= \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \cos\theta\,d\theta$$

$$= \sin\theta + C$$

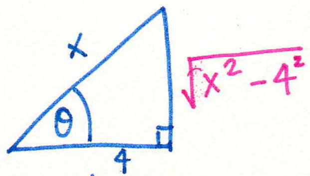
$$= \frac{\sqrt{x^2-4^2}}{x} + C \quad \#$$

แทนค่ากลับเป็น x

$x = 4\sec\theta$

$\frac{x}{4} = \sec\theta$

$\frac{x}{4} = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{4}{x}$



$$= \frac{\sqrt{x^2-16}}{4} + C$$

7.4 Trigonometric substitution การแทนค่าตรีโกณมิติ

We will be concerned with integrals that contain expressions of the form $\sqrt{a^2-u^2}$, $\sqrt{u^2 \pm a^2}$.

$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \le \pi$

(*)

Example 7.11 Evaluate $\int \frac{dx}{x^2\sqrt{4x^2-3}}$. $\boxed{\text{FORM 3}}$

$u^2 = 4x^2 \rightarrow u = 2x$

$a^2 = 3 \rightarrow a = \sqrt{3}$

$u = a\sec\theta \Rightarrow \boxed{2x = \sqrt{3}\sec\theta}$

$x = \frac{\sqrt{3}}{2}\sec\theta$

$\boxed{dx = \frac{\sqrt{3}}{2}\sec\theta\tan\theta\, d\theta}$

$$\int \frac{dx}{x^2\sqrt{4x^2-3}} = \int \frac{\frac{\sqrt{3}}{2}\sec\theta\tan\theta\, d\theta}{\left(\frac{\sqrt{3}}{2}\sec\theta\right)^2\sqrt{(\sqrt{3}\sec\theta)^2-3}}$$

$$= \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{2}{\sqrt{3}}\right)^2 \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\sqrt{3\sec^2\theta-3}}$$

$$= \left(\frac{\cancel{\sqrt{3}}}{\cancel{2}_1}\right)\cdot\left(\frac{\cancel{4}}{3}\right)\cdot\frac{1}{\cancel{\sqrt{3}}} \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\sqrt{\tan^2\theta}}$$

$$= \frac{2}{3}\int \frac{\cancel{\sec\theta\tan\theta}}{\sec^{\cancel{2}}\theta\cdot\cancel{\tan\theta}}\, d\theta$$

$$= \frac{2}{3}\int \cos\theta\, d\theta$$

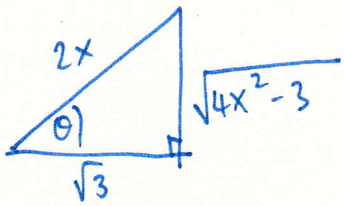
$$= \frac{2}{3}\sin\theta + C$$

$$= \frac{\cancel{2}}{3}\frac{\sqrt{4x^2-3}}{\cancel{2}x} + C$$

$$= \frac{1}{3}\frac{\sqrt{4x^2-3}}{x} + C. \#$$

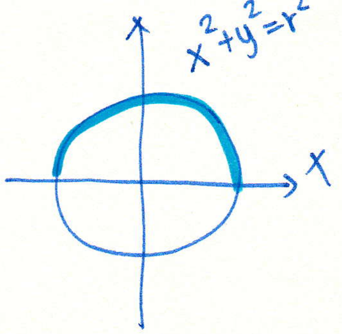
$2x = \sqrt{3}\sec\theta$

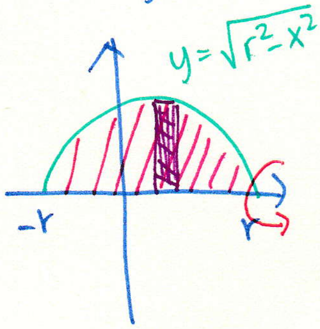
$\cos\theta = \frac{\sqrt{3}}{2x}$



ปัญหาท้าทายปัญญา

$\boxed{\text{ทรงกลม}}$ $V = \frac{4}{3}\pi r^3$





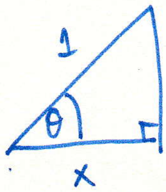
หมุนรอบแกน x ด้วยวิธี disk

$$V = \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

⋮

Ex $\int \frac{1}{\sqrt{1-x^2}}\,dx$

$$a=\sqrt{1-x^2}$$
$$a^2 = 1 - x^2$$
$$x^2+a^2=1$$

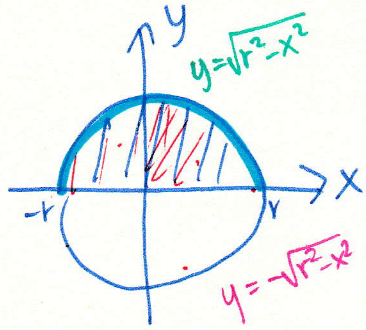


$$\left.\begin{aligned}\sin\theta &= a\\ \cos\theta &= x\\ \tan\theta &= \frac{a}{x}\end{aligned}\right.$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

ต่อแผ่น ๓ **พ.ท.วงกลม**



$$\text{พ.ท.วงกลม} = 2\int_{-r}^{r} \sqrt{r^2-x^2}\,dx$$

$$\boxed{A = 4\int_0^r \sqrt{r^2-x^2}\,dx}$$

$\cancel{x} = r\sin\theta$

⋮

$$\boxed{A=\pi r^2}$$