# แบบฝึกหัดทบทวน : ข้อ 1

$$\int \frac{dx}{(x^2-4)^{3/2}}$$

$u=x$    $a=2$

| ทบทวน | สมมติ |
| --- | --- |
| ① $\sqrt{a^2-u^2}$ ⇒ | $u = a\sin\theta$ |
| ② $\sqrt{a^2+u^2}$ ⇒ | $u = a\tan\theta$ |
| ③ $\sqrt{u^2-a^2}$ → | $u = a\sec\theta$ |

สมมติว่า $x = 2\sec\theta \rightarrow dx = 2\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{(x^2-4)^{3/2}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{[(2\sec\theta)^2-4]^{3/2}}$$

$$= 2\int \frac{\sec\theta\tan\theta\, d\theta}{[4(\sec^2\theta-1)]^{3/2}}$$

$$♡^{3/2} = (\sqrt{♡})^3$$

$\sec^2\theta - 1 = ?$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \cancel{\frac{2}{8}}\frac{1}{4} \int \frac{\sec\theta\tan\theta\, d\theta}{[\tan^2\theta]^{3/2}}$$

$$= \frac{1}{4}\int \frac{\sec\theta\, \cancel{\tan\theta}\, d\theta}{\tan^3\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\cot^2\theta\, d\theta = \cancel{\frac{1}{4}\int \frac{\sin^2\theta}{\cos^3\theta}\, d}$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{\overbrace{\cos\theta\, d\theta}^{du}}{\underbrace{\sin^2\theta}_{u^2}}$$

→ ใช้ $u = \sin\theta$
$du = \cos\theta\, d\theta$

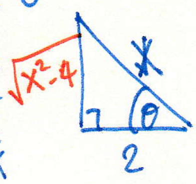
$$= \frac{1}{4}\int \frac{1}{u^2}\, du = -\frac{1}{4}\cdot\frac{1}{u} + C$$

$$= -\frac{1}{4}\cdot\frac{1}{\sin\theta} + C \quad —— (*)$$

จะเห็นว่า $x = 2\sec\theta$

$\sec\theta = \frac{x}{2}$

$\cos\theta = \frac{2}{x}$




จาก $\sin\theta = \frac{\sqrt{x^2-4}}{x}$

$$\therefore (*) = -\frac{1}{4}\cdot\frac{x}{\sqrt{x^2-4}} + C \quad \#$$

**Example 7.13** Evaluate $\int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}\,dx.$ du

สมมติลอง $u = e^{2x}+e^x+1$
$du = (2e^{2x}+e^x)\,dx$ ✗

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เลือก $a = e^x$ ; $da = e^x\,dx$

$$\therefore \int \frac{da}{\sqrt{a^2+a+1}} = \int \frac{da}{\sqrt{\left[a^2+2(a)\frac{1}{2}+\left(\frac{1}{2}\right)^2\right]-\left(\frac{1}{2}\right)^2+1}}$$

$$= \int \frac{da}{\sqrt{\left[a^2+2(a)\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\right]+\frac{3}{4}}} = \int \frac{da}{\sqrt{\left(a+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$u = a+\frac{1}{2}$

$a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

สมมติ $a+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$

$\therefore da = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

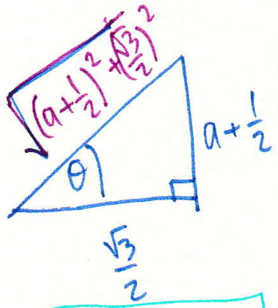
$$\int \frac{da}{\sqrt{\left(a+\frac{1}{2}\right)^2+\frac{3}{4}}} = \frac{\sqrt{3}}{2}\int \frac{\sec^2\theta\,d\theta}{\sqrt{\left(\frac{\sqrt{3}}{2}\tan\theta\right)^2+\frac{3}{4}}} = \frac{\sqrt{3}}{2}\int \frac{\sec^2\theta\,d\theta}{\sqrt{\frac{3}{4}\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}}}$$

$$= \frac{\cancel{\frac{\sqrt{3}}{2}}}{\cancel{\frac{\sqrt{3}}{2}}}\int \frac{\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec\theta}\,d\theta}$$

Trick $\int \sec\theta\,d\theta$ (ลองใช้ by parts)

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|+C$$




$$= \ln\left|\frac{\sqrt{\left(a+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}{\frac{\sqrt{3}}{2}}+\frac{a+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|+C$$

$$= \ln\left|\frac{\sqrt{\left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}}}{\frac{\sqrt{3}}{2}}+\frac{e^x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|+C$$

$$= \ln\left|\frac{2\sqrt{\left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}}}{\sqrt{3}}+\frac{2e^x+1}{\sqrt{3}}\right|+C \quad \#$$

$\tan\theta = ♡$
$\theta = \arctan ♡$

$= \theta + \sin\theta +$

## ④ การอินทิเกรตแยกเศษส่วนย่อย (partial fraction)

↳ $\boxed{\text{ฟังก์ชันตรรกยะ}}$ $\quad f(x) = \dfrac{p(x)}{q(x)}$ $\;p(x) \rightarrow$ ฟังก์ชันพหุนาม, $\;q(x) \rightarrow$ ฟังก์ชันพหุนาม

proper

**Ex** $\boxed{f(x) = \dfrac{x^2+4}{x^3(x+1)} \quad \begin{matrix} \deg(p(x)) = 2 \\ \deg(q(x)) = 4 \end{matrix} \Bigg\} \deg(p(x)) < \deg(q(x))}$

ฟังก์ชันตรรกยะแท้ (proper rational function)

$$g(x) = \frac{x^5+x^3+1}{x^2(x-1)(x+1)} \quad \begin{matrix} \deg(p(x)) = 5 \\ \deg(q(x)) = 4 \end{matrix} \Bigg\} \deg(p(x)) > \deg(q(x))$$

$$h(x) = \frac{x^2+2x-1}{(x-1)(x+1)} \quad \begin{matrix} \deg(p(x)) = 2 \\ \deg(q(x)) = 2 \end{matrix} \Bigg\} \deg(p(x)) = \deg(q(x)).$$

- สำหรับฟังก์ชันตรรกยะที่เป็นฟังก์ชันตรรกยะแท้ เราจะทำฟังก์ชันเป็นเศษส่วนย่อย (partial fraction)

- ถ้าฟังก์ชันตรรกยะไม่เป็นฟังก์ชันตรรกยะแท้ เราต้องทำฟังก์ชันให้เป็นฟังก์ชันตรรกยะแท้ก่อน.

**Ex** พหุนาม $\underbrace{3x^4+3x^3-5x^2+x-1}$ หารด้วย $x^2+x-2$

$$f(x) = \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}$$

(ตัวตั้ง = ตัวหาร × ผลหาร + เศษ)

$$\begin{array}{r|l} & 3x^2 \qquad\qquad +1 \quad \text{ผลหาร} \\ \hline x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\ & \mp 3x^4 \pm 3x^3 \mp 6x^2 \\ \hline & \qquad\qquad x^2+x-1 \\ & \qquad\qquad \pm x^2 \pm x \mp 2 \\ \hline & \qquad\qquad\qquad 1 \quad \text{เศษ} \end{array}$$

$$\therefore \frac{3x^4+3x^3-5x^2+x+1}{x^2+x-2} = 3x^2+1+\frac{1}{x^2+x-2}$$

ต้องการอินทิเกรต : $\displaystyle\int \frac{3x^4+3x^3-5x^2+x+1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{1}{x^2+x-2}\,dx$

## 7.5 Integrating rational functions by partial fractions

Recall that a rational function is a function that can be written as a quotient of two polynomials. Assume that $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where $P(x)$ and $Q(x)$ are polynomials. If $\deg P(x) < \deg Q(x)$, then $f(x)$ is said to be ***proper*** . If $\deg P(x) \geq \deg Q(x)$, then $f(x)$ is said to be ***improper*** .

We now find the form of partial fraction decomposition of a proper rational function $f(x) = \dfrac{P(x)}{Q(x)}$. Elementary algebra tells us that $Q(x)$ has only two types of irreducible factors which are of degree 1 or degree 2. Therefore, the partial fraction decomposition of $f(x)$ can be determined by using the following rules, **Linear factor and Quadratic factor rules.**

**Linear factor rule:** For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of $m$ partial fractions:
$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ldots + \dfrac{A_m}{(ax+b)^m}$, where $A_i \quad (i = 1, 2, \ldots, m)$ are constants.

## FORM 1

$\boxed{\dfrac{p(x)}{q(x)}}$ => สมมติว่า $\boxed{\dfrac{3}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}}$

หาว่า A, B คือใคร(ที่รอดลงมั้ย)

จาก $\dfrac{3}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$$3 = \frac{A\cancel{(x-1)}(x-2)}{\cancel{x-1}} + \frac{B(x-1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$3 = A(x-2) + B(x-1)$$

$$3 = Ax - 2A + Bx - B$$

$$3 = (A+B)x + (-2A-B)$$

เทียบ สปส $\left.\begin{aligned} A+B &= 0 \\ -2A+B &= 3 \end{aligned}\right\} A, B$

⊛ $\deg(p(x)) < \deg(q(x))$

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-a_1)(x-a_2)(x-a_3)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$$

แล้วหาสัมประสิทธิ์ $A_1, \dots, A_n$ จากการเทียบกับอีกฝั่งนะปร. $(a_1 \neq a_2 \neq a_3 \dots)$

ถ้ามีตัวประกอบบางตัวที่ซ้ำบางตัว

$$\frac{p(x)}{q(x)} = \frac{p(x)}{\boxed{(x-a_1)^2}\boxed{(x-a_2)^3}\boxed{(x-a_3)\dots(x-a_n)}} = \boxed{\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2}} + \boxed{\frac{B_1}{(x-a_2)} + \frac{B_2}{(x-a_2)^2} + \frac{B_3}{(x-a_2)^3}}$$

$$+ \boxed{\frac{C_3}{x-a_3} + \frac{C_4}{x-a_4} + \frac{C_5}{x-a_5} + \dots + \frac{C_n}{x-a_n}}$$

**Example 7.14** Evaluate $\int \frac{dx}{x^2+x-2}$.

$$\int \frac{dx}{x^2+x-2} = \int \frac{1}{(x+2)(x-1)}\,dx$$

แยกเศษส่วนย่อย
$$\frac{1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$
$$1 = A(x-1) + B(x+2)$$
$$1 = Ax - A + Bx + 2B$$
$$1 = (A+B)x + (-A+2B)$$

เทียบ สปส $\therefore$ $A+B = 0$ —①

$-A+2B = 1$ —②

①+②; $3B = 1 \Rightarrow B = \frac{1}{3}$

ได้ว่า $A = -\frac{1}{3}$

$$\therefore \frac{1}{(x+2)(x-1)} = -\frac{1}{3}\cdot\frac{1}{x+2} + \frac{1}{3}\cdot\frac{1}{x-1}$$

$$\therefore \int \frac{1}{(x+2)(x-1)}\,dx = \int \left(-\frac{1}{3}\cdot\frac{1}{x+2} + \frac{1}{3}\cdot\frac{1}{x-1}\right)dx$$
$$= -\frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int\frac{1}{x-1}\,dx$$
$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C \quad \#$$
$$= \ln\left|\frac{x-1}{x+2}\right|^{\frac{1}{3}} + C$$

พิจารณา $\int \frac{1}{x+2}\,dx$

$u = x+2$

$du = dx$

$$\therefore \int \frac{1}{x+2}\,dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + c$$
$$= \ln|x+2| + c$$

**Example 7.15** Evaluate $\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx$.

พิจารณา $\dfrac{2x^2-3x+4}{(x+1)(x-2)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$$2x^2-3x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$
$$2x^2-3x+4 = A(x^2-4x+4) + B(x^2-x-2) + C(x+1)$$
$$2x^2-3x+4 = (A+B)x^2 + (-4A-B+C)x + (4A-2B+C)$$

เทียบ สปส.:
$$\left.\begin{aligned} A+B &= 2 \\ -4A-B+C &= -3 \\ 4A-2B+C &= 4 \end{aligned}\right\} \longrightarrow \boxed{A=1,\ B=1,\ C=2}$$

$$\therefore \frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

$$\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx = \int\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}\right)dx$$
$$= \int\frac{1}{x+1}dx + \int\frac{1}{x-2}dx + \int\frac{2}{(x-2)^2}dx$$

$u = x-2$ ; $du = dx$

$$= 2\int\frac{1}{u^2}du = -\frac{2}{u} + C$$
$$= -\frac{2}{x-2} + C$$

$$= \ln|x+1| + \ln|x-2| - \frac{2}{x-2} + C \quad \#$$

$\boxed{\text{FORM 2}}$

$$\frac{p(x)}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$\frac{p(x)}{(x^2-4)(x+3)} = \frac{Ax+B}{x^2-4} + \frac{C}{x+3}$$ $\leftarrow$ $x^2-4$ แยก ตปก. ได้อีก

(*) $q(x)$ ต้อง แยกตัวประกอบให้สุด

$$\frac{p(x)}{\boxed{(x^2+1)^2}\,\boxed{x}\,\boxed{(x-1)}} = \boxed{\frac{A}{x}} + \boxed{\frac{B}{x-1}} + \boxed{\frac{Cx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2}}$$

**Quadratic factor rule :** For each factor of the form $(ax^2+bx+c)^m$ with $b^2-4ac<0$,

the partial fraction decomposition contains the following sum of $m$ partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_mx+B_m}{(ax^2+bx+c)^m},$$ where $A_i, B_i$ $(i=1,2,\ldots,m)$ are constants.

**Example 7.16** Evaluate $\displaystyle\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx$.

พิจารณา $$\frac{3x^2+x-2}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$3x^2+x-2=A(x^2+1)+(Bx+C)(x-1)$$

$$3x^2+x-2=A(x^2+1)+(Bx^2-Bx+Cx-C)$$

$$3x^2+x-2=(A+B)x^2+(-B+C)x+(A-C)$$

$$\therefore\ \left.\begin{aligned} A+B&=3\\ -B+C&=1\\ A\quad -C&=-2\end{aligned}\right\}$$

$$\boxed{A=1,\ B=2,\ C=3}$$

$$\int\frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx=\int\left(\frac{1}{x-1}+\frac{2x+3}{x^2+1}\right)dx$$

$$=\int\frac{1}{x-1}\,dx+\int\left(\frac{2x+3}{x^2+1}\right)dx \quad (*)$$

พิจารณา $$\int\frac{2x+3}{x^2+1}\,dx=\int\frac{\overbrace{2x\,dx}^{du}}{\underbrace{x^2+1}_{u}}+\int\frac{3}{x^2+1}\,dx$$

$u=x^2+1$ ; $du=2x\,dx$

$$\int\frac{\overbrace{2x\,dx}^{du}}{\underbrace{x^2+1}_{u}}=\int\frac{1}{u}\,du=\ln|u|+C$$

$$=\ln|x^2+1|+C.$$

$$(*)=\ln|x-1|+\ln|x^2+1|+3\arctan x+C.\ \#$$

**Example 7.17** Evaluate $\displaystyle\int \frac{x+4}{x^2(x^2+4)}\,dx$.

$$\frac{x+4}{x^2(x^2+4)} = \boxed{\frac{A}{x} + \frac{B}{x^2}} + \boxed{\frac{Cx+D}{x^2+4}}$$

$\left(A = \frac{1}{4},\ B = 1,\ C = -\frac{1}{4},\ D = -1\right)$

**Example 7.18** Evaluate $\int \frac{x^3 - 4x}{(x^2+1)^2}\,dx$.

$$\frac{x^3-4x}{\boxed{(x^2+1)^2}} = \boxed{\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}}$$

$A = 1$, $B = 0$, $C = -5$, $D = 0$

### Integrating improper rational functions

**Example 7.19** Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$.

The integrand can be expressed as

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$$