สมการเชิงอนุพันธ์

- ① order
- ② หาปินคำตอบ
  - ① คำตอบทั่วไป (general solution)
  - ② คำตอบเฉพาะ (particular solution)

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① Separable equation $h(y)\dfrac{dy}{dx} = g(x)$

② Linear equation (First order)

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

$y = mx + b$

* สปส หน้า $\dfrac{dy}{dx}$ ต้องเป็น 1.

8.2 First-order Linear Equations

First-order linear equations are differential equations that can be expressed in the form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{8.10}$$

Generally, we look for a general solution on some common interval where $p(x)$ and $q(x)$ are both continuous.

Notice from (8.10) that linear equations have $\frac{dy}{dx}$ and y both of degree one. If a first-order equation has this property, then it can be rewritten as (8.10).

$$\boxed{\frac{dy}{dx}} + p(x)\boxed{y} = q(x)$$

Example 8.6

Original equation	Linear form	$p(x)$	$q(x)$
$\boxed{\frac{dy}{dx}} + x^2\boxed{y} = e^x$	$\boxed{\frac{dy}{dx}} + x^2\boxed{y} = e^x$	x^2	e^x
$y' - 3e^x y = e^x$	$\boxed{y'} - 3e^x\boxed{y} = e^x$	$-3e^x$	e^x
$x\boxed{\frac{dy}{dx}} = 1$	$\boxed{\frac{dy}{dx}} = \frac{1}{x}$	0	$\frac{1}{x}$
$x\boxed{y'} + 2\boxed{y} = 4x^2$	$\boxed{y'} + \frac{2}{x}\boxed{y} = \frac{4x^2}{x}$	$\frac{2}{x}$	$4x$
$\frac{dy}{dx} + y - \frac{1}{1+e^x} = 0$	$\boxed{\frac{dy}{dx}} + \boxed{y} = \frac{1}{1+e^x}$	1	$\frac{1}{1+e^x}$

□

แบบฝึกหัดทบทวน คาบเรียนวันจันทร์ที่ 19 พฤศจิกายน พ.ศ.2561

ชื่อ-สกุล.. รหัสนักศึกษา.............................. ลำดับที่...........

1. ~~จงตรวจสอบว่า $y = e^{3x}$ เป็นคำตอบของสมการเชิงอนุพันธ์ $y'' - y' - 6y = 0$ หรือไม่~~

$\boxed{\text{Sep}}$

$$(x^2+1)\frac{dy}{dx} = yx - y$$

$$(x^2+1)\boxed{\frac{dy}{dx}} = y(x-1) \rightarrow h(y)\boxed{\frac{dy}{dx}} = g(x)$$

$$\boxed{\frac{1}{y}}\boxed{\frac{dy}{dx}} = \boxed{\frac{x-1}{x^2+1}}$$

2. จงเขียนสมการเชิงอนุพันธ์ $(x^2+1)\frac{dy}{dx} = yx - y$ ให้อยู่ในรูปแบบที่กำหนดให้

(a) Separable form แบบแยกตัวแปรได้ $h(y)\dfrac{dy}{dx} = g(x)$

$h(y) =$ $\frac{1}{y}$ $\qquad g(x) =$ $\frac{x-1}{x^2+1}$

(b) Linear form แบบเชิงเส้น $\dfrac{dy}{dx} + p(x)y = q(x)$

$p(x) =$ $-\frac{x-1}{x^2+1}$ $\qquad q(x) =$ 0

3. ~~จงแก้สมการเชิงอนุพันธ์ $y - \cos^2 x \dfrac{dy}{dx} = 0$~~

$\boxed{\text{Linear}}$

$$(x^2+1)\frac{dy}{dx} = yx - y$$

$$(x^2+1)\frac{dy}{dx} = y(x-1)$$

$$\boxed{\frac{dy}{dx}} + p(x)\boxed{y} = q(x)$$

$$(x^2+1)\frac{dy}{dx} - y(x-1) = 0$$

$$\boxed{\frac{dy}{dx}} - \frac{x-1}{x^2+1}\boxed{y} = 0$$

$$h(y)\frac{dy}{dx} = g(x)$$ Separable eq$^{\underline{n}}$

$$\downarrow$$

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

$$\downarrow$$

$\boxed{\text{ความพยายาม}}$ อยากให้เทอมอยู่ในรูป → เอา $\mu(x)$ ไปคูณเพื่อให้เป็นรูป

$$\frac{d}{dx}[\mu(x)\cdot y] = \mu(x)q(x) \quad \text{———} ⊛$$

⊛ $\mu(x)$ ควรเป็นอะไรถึงจะอยู่ในฟอร์ม ⊛

จาก $$\frac{dy}{dx} + p(x)y = q(x)$$

คูณทั้งสมการด้วย $\mu(x)$ ทั้งสองข้าง

$$\underline{\mu(x)\frac{dy}{dx}} + \mu(x)p(x)y = \mu(x)q(x)$$

อยากได้ $$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}[\mu(x)\cdot y]$$

$$\downarrow \text{Diff ผลคูณ}$$

$$\underline{\mu(x)\frac{dy}{dx}} + y\frac{d}{dx}\mu(x)$$

เทียบระหว่าง $$\mu(x)p(x)\underline{y} = \underline{y}\frac{d\cancel{x}\mu(x)}{dx}$$

$$\downarrow$$

$$\mu(x)p(x) = \frac{d}{dx}\mu(x)$$ หาว่า $\mu(x)$ คือใคร

$$\mu(x)p(x)\,dx = d\mu(x)$$

$$p(x)\,dx = \frac{1}{\mu(x)}d\mu(x)$$

$$\int p(x)\,dx = \int\frac{1}{\mu(x)}d\mu(x)$$

$$\int p(x)\,dx = \ln|\mu(x)|$$

$$\boxed{\mu(x)^{*} = e^{\int p(x)\,dx}.}$$

*ตัวประกอบการอินทิเกรต (Integrating Factor)

DEFINITION Given a linear equation (8.10), an **integrating factor** for (8.10) is defined by

$$\mu = e^{P(x)}$$

for some antiderivative $P(x)$ of $p(x)$.

Remark. A linear equation has infinitely many integrating factors, due to the constant of integration from $\int p(x)dx$. So, different integrating factors only differ by their coefficients. Therefore, a general solution obtained by this method is independent of the choice of integrating factors. We may write

$$\mu = e^{\int p(x)dx}$$

Below is a summary of how to apply the Method of Integrating Factors to the equation (8.10). **If the original equation is not in the form of (8.10), it needs to be rewritten first!**

Step 1. Calculate $\int p(x)dx$ and choose an integrating factor

$$\mu = e^{P(x)} = e^{\int p(x)dx}$$

for the linear equation. You may take the constant of integration to be zero to make things simpler.

Step 2. Multiply both sides of (8.10) by μ and express the result as

$$\frac{d}{dx}(\mu y) = \mu q(x).$$

Step 3. Integrate both sides of the result from Step 2 and then solve for y.

สรุป

① $\dfrac{dy}{dx} + \boxed{p(x)}\, y = q(x)$ —— ⊛

② หา integrating factor $\mu(x) = e^{\int \boxed{p(x)}\, dx}$

③ เอา $\mu(x)$ ไปคูณ ⊛ ทั้งสองข้าง

④ $$\frac{d}{dx}[\mu(x) y] = \mu(x) q(x)$$

⑤ แก้แบบ Separable

$$d(\mu(x) y) = \mu(x) q(x)\, dx$$

⑥ integrate

$$\int d(\mu(x) y) = \int \mu(x) q(x)\, dx$$

$$\mu(x) y = \int \mu(x) q(x)\, dx$$

$$\boxed{y = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx}$$

Example 8.7 Solve the differential equation

$$\boxed{\frac{dy}{dx}} - \boxed{y} = e^{2x}. \qquad \frac{dy}{dx} + p(x)y = q(x)$$

$p(x) = -1$

$q(x) = e^{2x}$

• หา integrating factor $\boxed{\mu(x) = e^{\int p(x)dx}}$

$$\mu(x) = e^{\int -1\,dx} = e^{-x}$$

• คูณสมการด้วย integrating factor ทั้งสองข้าง

$$e^{-x}\left[\frac{dy}{dx} - y\right] = e^{-x}e^{2x}$$

$$\frac{d}{dx}[e^{-x}y] = e^{x} \qquad \left(\frac{d}{dx}(\mu(x)y) = \mu(x)q(x)\right)$$

$$d(e^{-x}y) = e^{x}dx$$

$$\int d(e^{-x}y) = \int e^{x}dx$$

$$e^{-x}y = e^{x} + C$$

บวก C ทันที ที่อินทิเกรตเสร็จ (*)

$$\boxed{\therefore y = e^{2x} + Ce^{x}}$$

Problem 8.5 Solve the differential equation

$$x\boxed{\frac{dy}{dx}} + 2\boxed{y} = x^2 - x + 1.$$

$$\therefore \boxed{\frac{dy}{dx}} + \underbrace{\frac{2}{x}}_{p(x)}\boxed{y} = \underbrace{x - 1 + \frac{1}{x}}_{q(x)}$$

• หา integrating factor. $\mu(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx}$

$$= e^{2\int \frac{1}{x}dx} = e^{2\ln|x|}$$

$$= e^{\ln|x|^2} = x^2$$

$$\boxed{\mu(x) = x^2}$$

• คูณสมการทั้งสองข้างด้วย $\mu(x)$

$$x^2\left[\frac{dy}{dx} + \frac{2}{x}y\right] = x^2\left[x - 1 + \frac{1}{x}\right]$$

$$\frac{d}{dx}[x^2y] = x^3 - x^2 + x$$

$$d(x^2y) = (x^3 - x^2 + x)dx$$

$$\int d(x^2y) = \int (x^3 - x^2 + x)dx$$

$$x^2y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\boxed{y = \frac{x^2}{2} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}}$$

ปัญหาค่าเริ่มต้น

Problem 8.6 Solve the initial-value problem

$$x\frac{dy}{dx} - y = x, \quad y(1) = 2.$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1$$

$$p(x) = -\frac{1}{x}$$

$$q(x) = 1$$

• หา integrating factor : $\mu(x) = e^{\int p(x)\,dx}$

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|}$$

$$= e^{\ln|x|^{-1}} = \frac{1}{x} \qquad \boxed{\mu(x) = \frac{1}{x}}$$

• คูณสมการด้วย $\mu(x)$

$$\mu(x)\left[\frac{dy}{dx} - \frac{1}{x}y\right] = \mu(x)\cdot 1$$

$$\frac{1}{x}\left[\frac{dy}{dx} - \frac{1}{x}y\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{1}{x}$$

$$d\left(\frac{1}{x}y\right) = \frac{1}{x}dx$$

$$\int d\left(\frac{1}{x}y\right) = \int \frac{1}{x}dx$$

$$\frac{1}{x}y = \ln|x| + C$$

$$\boxed{y = x\ln|x| + xC}$$

$y(1) = 2$

$$2 = 1\ln|1| + 1\cdot C$$

$$\boxed{C = 2}$$

คำตอบเฉพาะ คือ $\boxed{y = x\ln|x| + 2x}$ ⊛

8.3 Applications

This section provides some applications of first-order differential equations we have learned in previous sections.

- **Exponential Growth and Decay Models** แบบจำลอง การเพิ่มขึ้นหรือเสื่อมสลายแบบ exponential

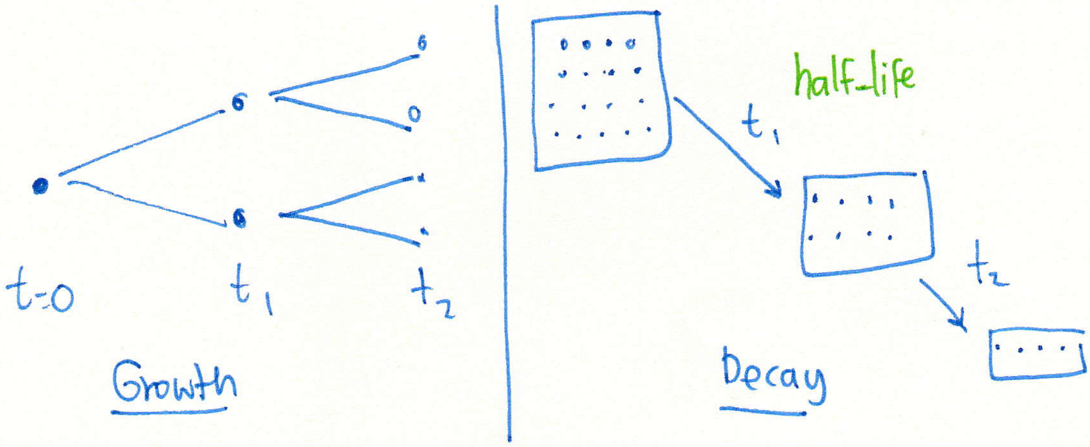
DEFINITION A quantity (ปริมาณ) $y = y(t)$ is said to have an **exponential growth model** if it increases at a rate that is propotional to the amount of the quantity present, and it is said to have an **exponential decay model** if it decreases at a rate that is propotional to the amount of the quantity present. Thus, for the former case, the quantity $y(t)$ satisfies an equation of the form

$$\boxed{\frac{dy}{dt} = ky \quad (k > 0)} \quad \text{Growth} \tag{8.12}$$

and for the latter, the quantity $y(t)$ satisfies an equation of the form

$$\boxed{\frac{dy}{dt} = -ky \quad (k > 0)} \quad \text{Decay} \tag{8.13}$$

The positive constant k is called the **growth constant** or the **decay constant**, as appropriate.



Let us imagine that we are studying the behaviour of a function $\boxed{y = y(t),}$ which changes over time t. Suppose experimentation shows that $y(t)$ satisfies a differential equation (8.12)

$$\boxed{\frac{dy}{dt}} = ky \quad (k > 0).$$

อัตราการเปลี่ยนแปลง เป็น k เท่า ของปริมาณเดิม.

We now need to solve this equation to study the behaviour of $y(t)$.

อยากรู้ว่า
ปริมาณ ณ เวลา t เป็นเท่าไร

หา $\boxed{y(t)\ ?}$

Problem 8.7 Solve the initial-value problem

$$\boxed{\frac{dy}{dt} = ky,} \quad y(0) = y_0$$

for $k > 0$.

หา $y(t) = ?$

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + C_1$$

$$\therefore\ y = e^{kt + C_1}$$

ได้ว่า $y = \boxed{e^{C_1}} \cdot e^{kt} \longrightarrow \boxed{y = Ce^{kt}}$

คำตอบทั่วไป

หาคำตอบเฉพาะ: $y(0) = y_0$

$$y_0 = Ce^{k(0)} \longrightarrow C = y_0$$

$$\therefore\ \boxed{y = y_0 e^{kt}}\ ;\ k > 0$$

บางครั้ง เราอาจจะต้องหาค่า k เอง!
จากสถานการณ์ที่กำหนดให้

Similarly, if $y(t)$ satisfies a differential equation (8.13) with initial condition $y(0) = y_0$, then

$$\boxed{y = y_0 e^{-kt}.} \quad \text{decay model} \tag{8.14}$$

It is a fact of physics that radioactive elements (e.g. plutonium, radium, uranium, etc.) disintegrate over time in the process called *radioactive decay*. Let $y(t)$ be the quantity of a radioactive element at time t. It is shown that $y(t)$ has an exponential decay model (8.13); that means it is explained by the rule (8.14)

$$y = y_0 e^{-kt}$$

where $k > 0$ is the decay constant and $y(0)$ is the initial quantity at time $t = 0$.

Problem 8.8 **Half-life** is the time required for a radioactive element to reduce its quantity by half.

Denote by T the half-life of a radioactive element. Use (8.14) to write T in terms of the decay constant k. If the half-life of radium-226 is 1600 years, find its decay constant.

$$\boxed{y_0} \xrightarrow{T} \boxed{\frac{y_0}{2}}$$

$y(t) = y_0 e^{-kt}$ → y_0 คือปริมาณเริ่มต้น.

$$\frac{dy}{dt} = -ky \;\Rightarrow\; y = y_0 e^{-kt} \quad ; \; y_0 \text{ คือ ปริมาณเริ่มต้น}$$

เมื่อเวลาผ่านไป T ปริมาณสาร ลดลงเหลือ $\frac{y_0}{2}$

$$\frac{y_0}{2} = y(T) = y_0 e^{-kT}$$

$$\therefore \frac{\cancel{y_0}}{2} = \cancel{y_0} e^{-kT}$$

$$\frac{1}{2} = e^{-kT}$$

$$\ln\left|\frac{1}{2}\right| = \ln e^{-kT}$$

$$\ln 2^{-1} = -kT \ln(e)$$

$$-\ln 2 = -kT$$

$$\boxed{k = \frac{\ln 2}{T}}$$

$$T = 1600$$

$$\therefore k = \frac{\ln 2}{1600}$$

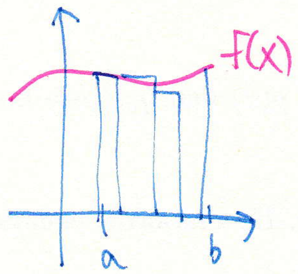
$$\boxed{y(t) = y_0 e^{-\frac{\ln 2}{1600} t}}$$

Review

$\boxed{\text{CH5}}$ Integrate เบื้องต้น

$\boxed{\text{หา พ.ท.ใต้กราฟ}}$ | $\boxed{\text{Antiderivative}}$

หา ฟังก์ชันอะไรบ้าง ที่ diff แล้วได้ตัวมันเอง



- Riemann Sum.

↓

$$A = \int_a^b f(x)\,dx$$

$$f(x) = \sin x$$

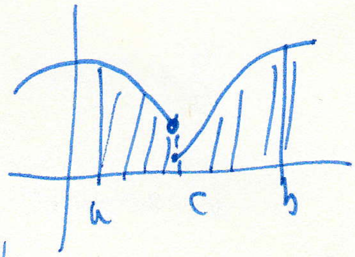
↓

$$\int f(x)\,dx = F(x) + C$$

$$F'(x) = f(x)$$

- เปลี่ยนตัวแปร.
- สูตรง่าย ๆ

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



FTC

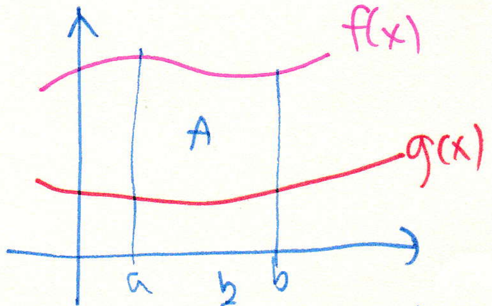
① $\int_a^b f(x)\,dx = F(b) - F(a)$

② $\frac{d}{dx}\int_a^x f(t)\,dt = \cancel{F}f(x)$

Ex $\boxed{\frac{d}{dx}\int_a^x e^{e^{e^t}}\,dt = e^{e^{e^x}}}$

$\boxed{\text{CH6}}$ Applications

① พ.ท.ใต้กราฟ.



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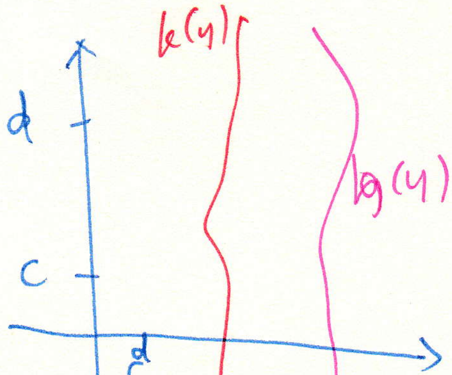
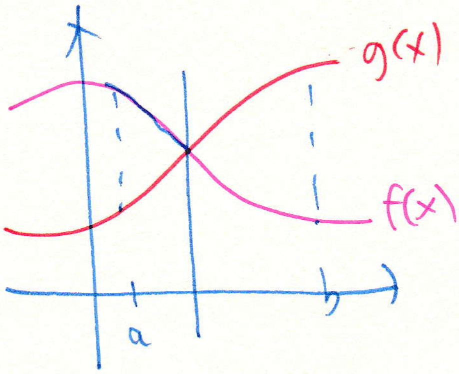


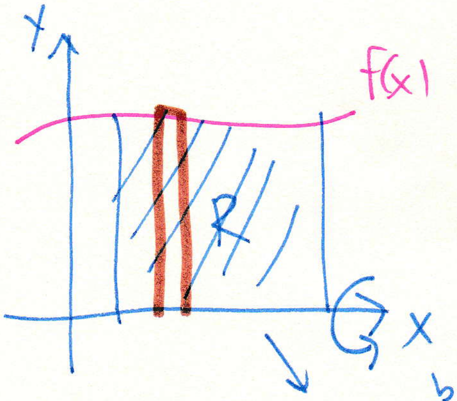
$$A = \int_a^b [f(x) - g(x)]\,dx$$

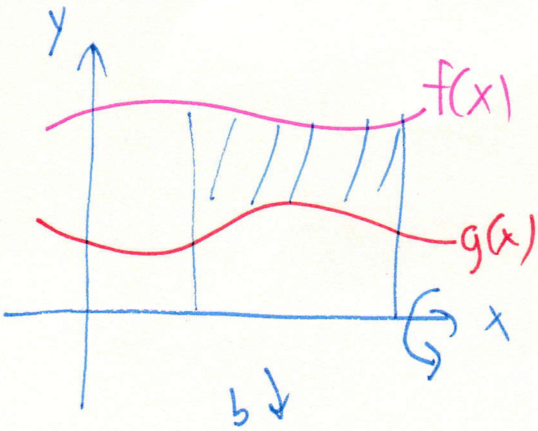
$$A = \int_c^d [h(y) - g(y)]\,dy$$



② ปริมาตร

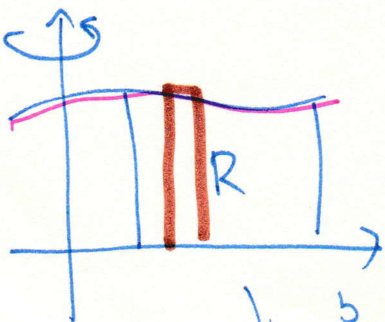
$\boxed{1}$ Disk / washers * สร้างแถบให้ตั้งฉากแกนหมุน





$$V = \int_a^b \pi [f(x)]^2 \, dx \qquad V = \int_a^b \pi \left[[f(x)]^2 - [g(x)]^2 \right] dx.$$

$\boxed{2}$ Cylindrical shell * สร้างแถบให้ขนานกับแกนหมุน



$$V = \int_a^b 2\pi x (f(x)) \, dx$$

CH7 เทคนิคการอินทิเกรต

1. by parts — u, dv ให้ดีๆ
2. ใช้เอกลักษณ์ตรีโกณ
3. แทนค่าตรีโกณ
   - $\sqrt{a^2-u^2} \rightarrow u = a\sin\theta$
   - $\sqrt{a^2+u^2} \rightarrow u = a\tan\theta$
   - $\sqrt{u^2-a^2} \rightarrow u = a\sec\theta$.
4. partial fraction → แยกตัวประกอบให้สุด แล้วสมมติ

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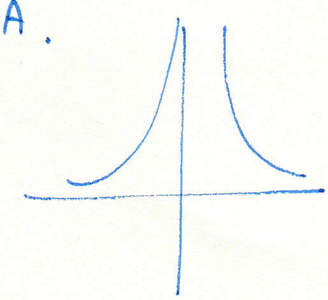
5. improper integral → Type1 ขอบเขตอนันต์ $-\infty$, $+\infty$
   - Type 2 มีจุดที่ฟังก์ชันไม่ต่อเนื่อง แบบมี V.A.



   - Type 3 ผสม

CH8 DEs.

→ การตรวจสอบการเป็นคำตอบ

→ สมการเชิงเส้น / แยกตัวแปร

→ แยกตัวแปรได้

→ Linear → $\mu(x) = e^{\int p(x)\,dx}$

→ Application.