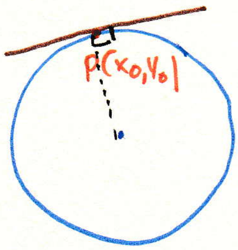


$f(x) = \dfrac{x^2-4}{x-2}$; โดเมนของฟังก์ชัน $f(x) = \dfrac{x^2-4}{x-2}$ คือ $\boxed{\mathbb{R} - \{2\}}$

- $f(0) = \dfrac{0^2-4}{0-2} = 2$
- $f(1) = \dfrac{1^2-4}{1-2} = 3$
- $f(2) = \boxed{X}$

$f(x)$ ที่ $x=a$

ลิมิตซ้าย $\lim\limits_{x\to a^-} f(x) = L$

ลิมิตขวา $\lim\limits_{x\to a^+} f(x) = L$

ฟังก์ชัน $f(x)$ มีลิมิต ที่ $x = a$ ถ้า $\lim\limits_{x\to a^+} f(x) = L = \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a} f(x)$

1

Limits and Continuity

เส้นสัมผัส

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on the graph of f, find an equation of the line that is tangent to the graph of f at P. (Figure 1.1)

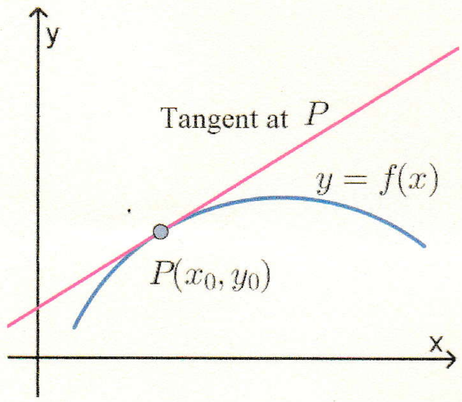


Figure 1.1: A picture of tangent line at point P

Example 1.1 Find an equation for the tangent line to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

Solution.

สมการเส้นตรง รู้จุด $\boxed{(x_0, y_0)}$, ความชัน m_{tan}

$$y - y_0 = m_{tan}(x - x_0)$$

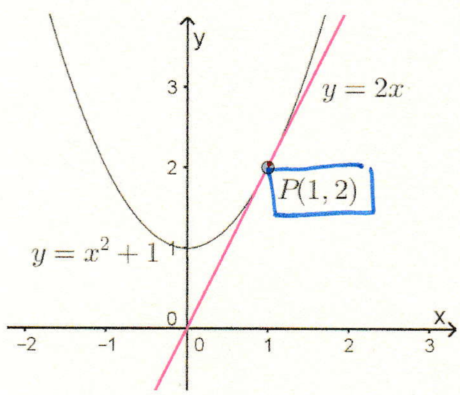
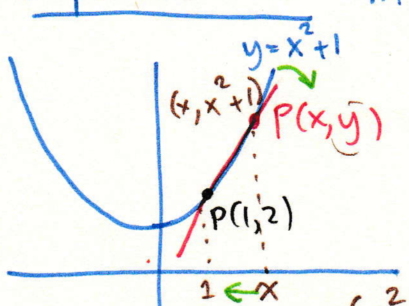
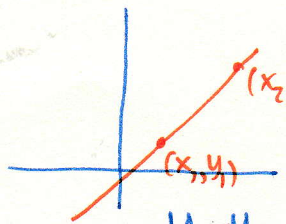


Figure 1.2: Graph of $y = x^2 + 1$

Slope ของเส้นตรง $m = \dfrac{\Delta y}{\Delta x}$





$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m_{sec} = \frac{(x^2+1) - 2}{x - 1}$$

ถ้าขยับให้ x เข้าใกล้ 1 จะได้ความชันของเส้นสัมผัสโค้งที่จุด (1,2)

1.1 Limits

Limits If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

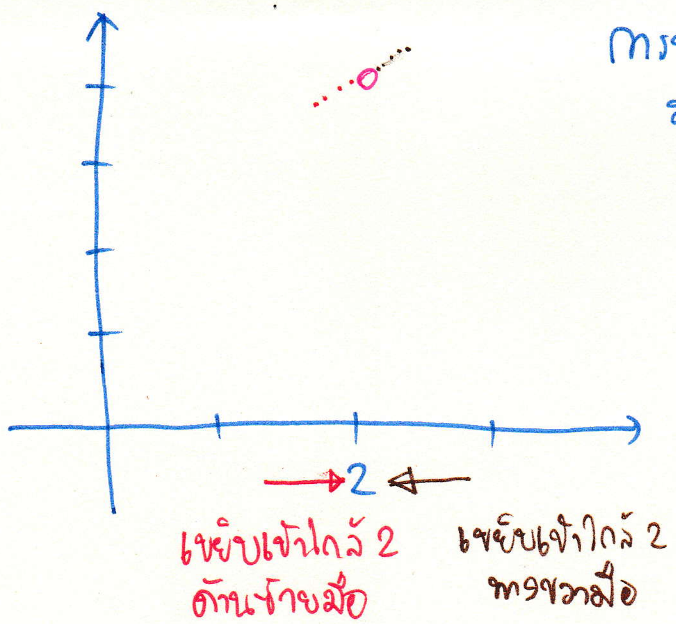
$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L", or "$f(x)$ approaches L as x approaches a".

Example 1.2 Use numerical evidence to make a conjecture about the value of $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution.

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	3.9999	4.0001	4.001	4.01	4.1



กรณีที่ $x \to 2$ ด้านบวก/ลบ
มีค่าของฟังก์ชันเข้าใกล้ 4

$\Downarrow$

$$\lim_{x \to 2} f(x) = 4$$

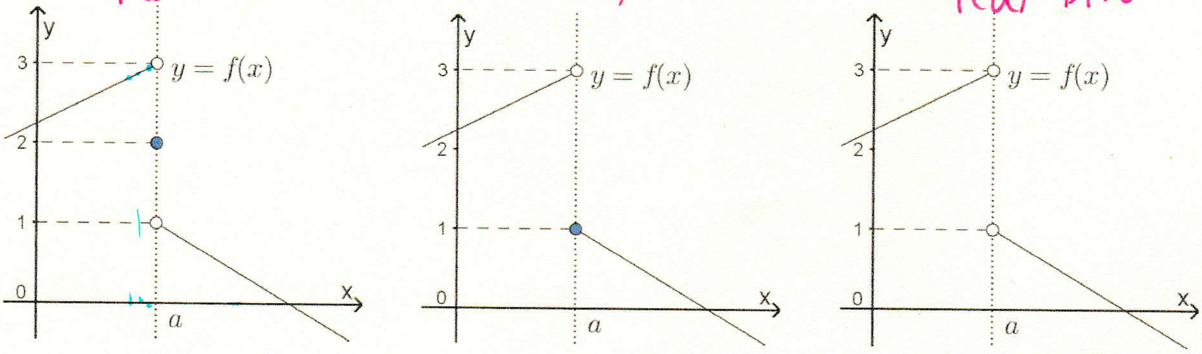
Example 1.4 For the functions in Figure 1.3 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.

$\lim_{x \to a^-} f(x) = 3$ $\lim_{x \to a^+} f(x) = 1$ $f(a) = 2$

$\lim_{x \to a^+} f(x) = 3$ $\lim_{x \to a^+} f(x) = 1$ $f(a) = 1$

$\lim_{x \to a^-} f(x) = 3$ $\lim_{x \to a^+} f(x) = 1$ $f(a)$ DNE



(a) $\lim_{x \to a} f(x) =$ DNE ↑ does not exist

(b) $\lim_{x \to a} f(x) =$ DNE

(c) $\lim_{x \to a} f(x) =$ DNE

Figure 1.3: A picture for Exercise 1.4 .

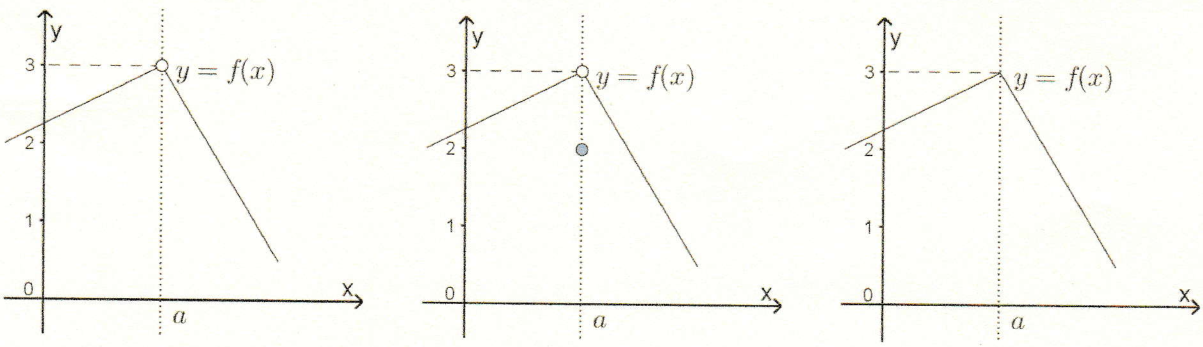
Example 1.5 For the functions in Figure 1.4 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.

$\lim_{x \to a^-} f(x) = 3$ $\lim_{x \to a^+} f(x) = 3$ $f(a)$ DNE

$\lim_{x \to a^-} f(x) = 3$ $\lim_{x \to a^+} f(x) = 3$ $f(a) = 2$

$\lim_{x \to a^-} f(x) = 3$ $\lim_{x \to a^+} f(x) = 3$ $f(a) = 3$



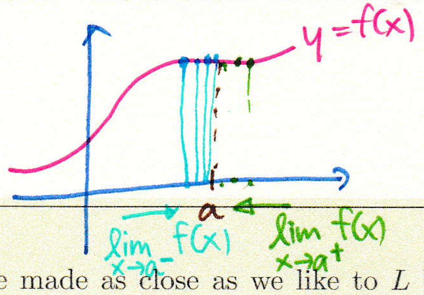
(a) $\lim_{x \to a} f(x) = 3$

(b) $\lim_{x \to a} f(x) = 3$

(c) $\lim_{x \to a} f(x) = 3$

Figure 1.4: A picture for Exercise 1.5

1.1.1 One-sided Limits



ONE-SIDED LIMITS If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\boxed{\lim_{x \to a^+} f(x) = L}$$ ลิมิตด้านขวา

("the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right".)

and if the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

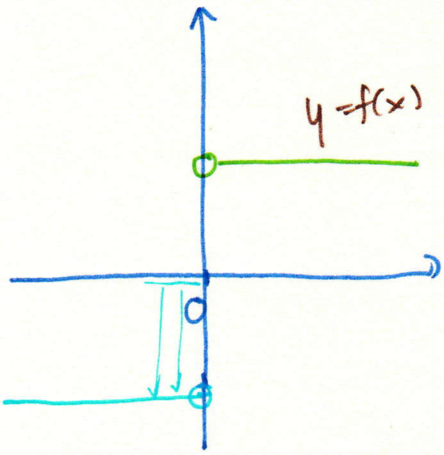
$$\boxed{\lim_{x \to a^-} f(x) = L}$$ ลิมิตทางซ้าย

("the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left".)

Example 1.3 Explain why $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution. พิจารณา $f(x) = \frac{|x|}{x}$ จะเห็นว่า $f(x) = \begin{cases} \frac{x}{x} & ; x > 0 \\ \frac{(-x)}{x} & ; x < 0 \end{cases}$

$\therefore f(x) = \begin{cases} 1 & ; x > 0 \\ -1 & ; x < 0 \end{cases}$



จะเห็นว่า $\lim_{x \to 0^-} f(x) = -1$

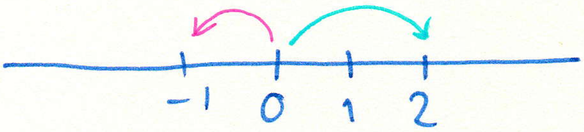
$\lim_{x \to 0^+} f(x) = 1$

เนื่องจาก $\lim_{x \to 0^-} f(x) = -1 \neq 1 = \lim_{x \to 0^+} f(x)$; $\lim_{x \to 0} f(x)$ D

THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at $x = a$ if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

$f(x) = |x|$ $|2| = 2$ $|-1| = 1$



$\downarrow$

peicewise-defined function

f(x) เขียนได้แบบของเงื่อนไข

$$f(x) = \begin{cases} \boxed{x \; ; \; x \geq 0} \\ \boxed{-(x) \; ; \; x < 0} \end{cases}$$

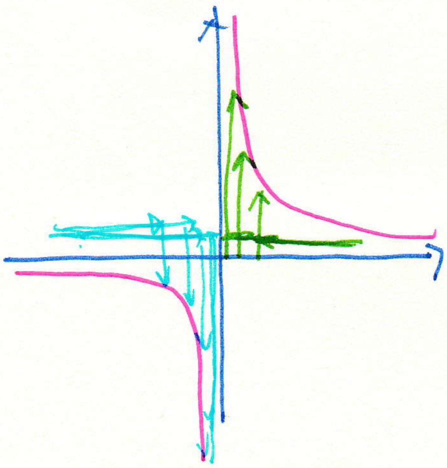
Ex $f(-1) = -(-1) = 1$

$f(100) = 100$

$f(x) = \dfrac{1}{x}$

f(0) หาค่าไม่ได้

$$\boxed{\lim_{x \to 0} f(x) = ?!?}$$



- ค่า x เข้าใกล้ 0 ทางบวก
  ค่าของ f(x) เพิ่มขึ้นมาก ๆ อย่างไม่มีขอบเขต.
- ค่า x เข้าใกล้ 0 ทางลบ
  ค่าของ f(x) ลดลงมาก ๆ อย่างไม่มีขอบเขต.

Left $\lim_{x \to 0^-} f(x) = -\infty$ Right $\lim_{x \to 0^+} f(x) = \boxed{+\infty}$

∞ (infinity) เป็นสัญลักษณ์ที่บอกการเพิ่มขึ้นหรือลดลงอย่างไม่มีขอบเขต.

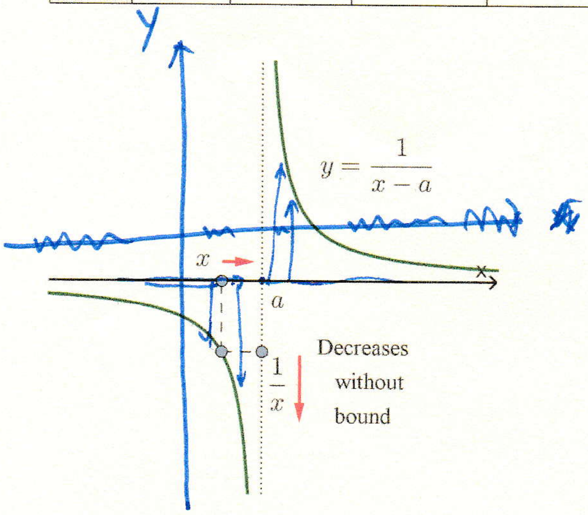
$\cancel{\infty - \infty = 0}$

1.1.2 Infinite Limits ลิมิตอนันต์

$f(x) = \frac{1}{x-a}$

Sometimes the values of the function increase or decrease without bound.

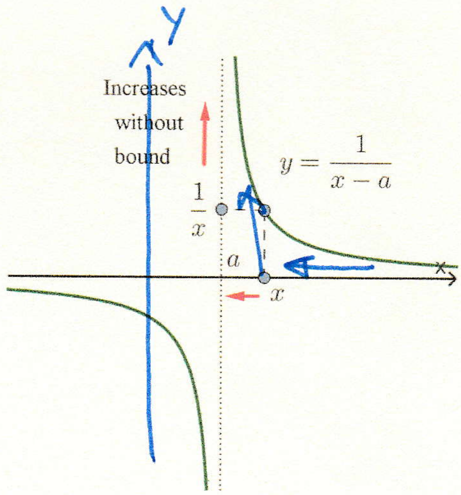
x	$a-1$	$a-0.1$	$a-0.01$	$a-0.001$	$a-0.0001$	$\cdots$	a
$\frac{1}{x-a}$	-1	-10	-100	-1000	-10,000	$\cdots$	



$$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

Figure 1.5: $\lim_{x \to a^-} \frac{1}{x-a} = -\infty$

x	a	$\cdots$	$a+0.0001$	$a+0.001$	$a+0.01$	$a+0.1$	$a+1$
$\frac{1}{x-a}$		$\cdots$	10,000	1000	100	10	1

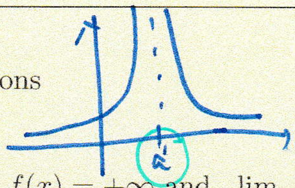


$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$$

Figure 1.6: $\lim_{x \to a^+} \frac{1}{x-a} = +\infty$

$\therefore \lim_{x \to a} f(x)$ DNE

Infinite Limits The expressions

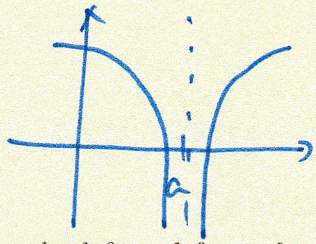


$$\lim_{x\to a^-} f(x) = +\infty \text{ and } \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\boxed{\lim_{x\to a} f(x) = +\infty}$$

Similarly, the expressions



$$\lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\boxed{\lim_{x\to a} f(x) = -\infty}$$

Example 1.6 For the functions in Figure 1.7 , describe the limits at $x = a$ in appropriate limit notation.

Solution.

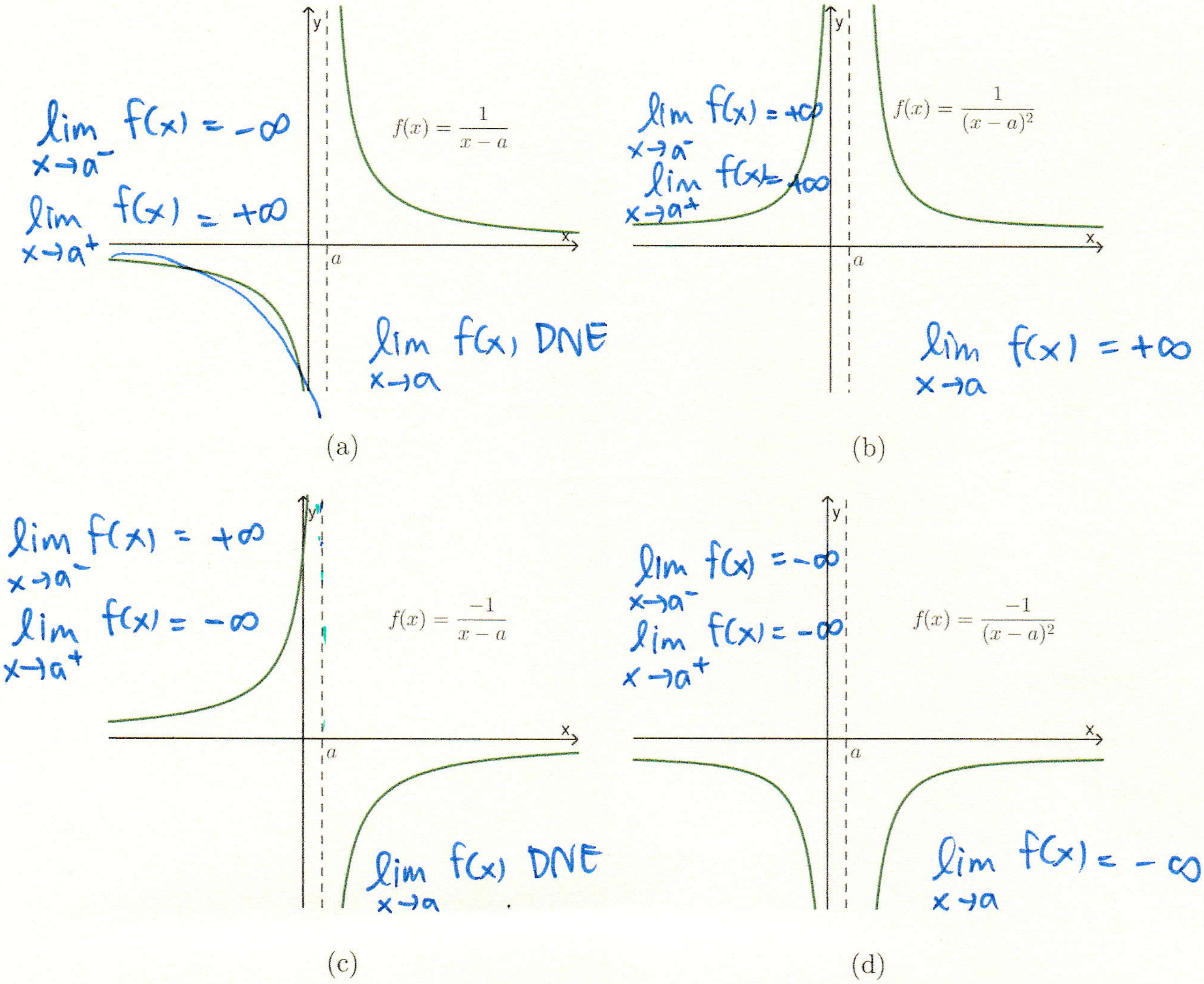
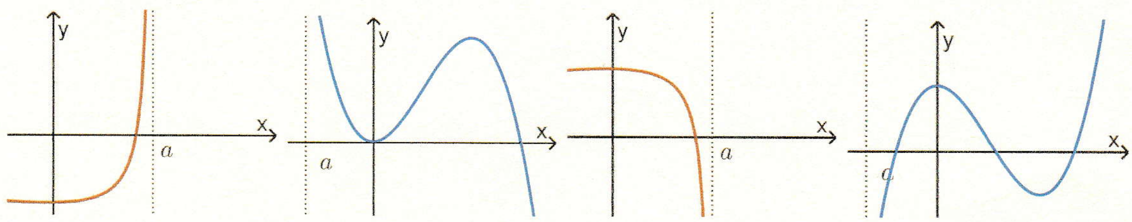


Figure 1.7: A picture for Exercise 1.6

1.1.3 Vertical Asymptotes เส้นกำกับแนวตั้ง | เส้นกำกับแนวยืน

If the graph of $f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit, we call the line $x = a$ ***vertical asymptote*** of the curve $y = f(x)$.

Figure 1.8 illustrates geometrically what happen when any of the following situations occur:



$\lim_{x\to a^-} f(x) = +\infty$ $\quad\lim_{x\to a^+} f(x) = +\infty$ $\quad\lim_{x\to a^-} f(x) = -\infty$ $\quad\lim_{x\to a^+} f(x) = -\infty$

Figure 1.8: Examples of vertical asymptotes

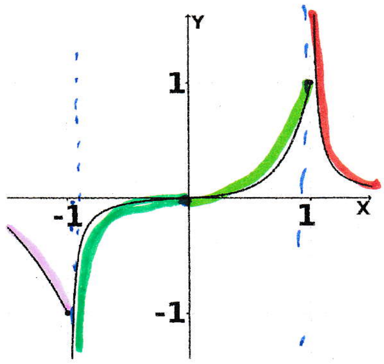
จะเรียก $x = a$ ว่าเป็นเส้นกำกับแนวยืน (VA) ถ้า $\lim_{x\to a^-} f(x) = +/-\infty$ (หรือ $x \to a^+$ หรือ $x \to a$)

แบบทดสอบย่อย

เพื่อเช็คชื่อเข้าชั้นเรียน ประจำวันอังคารที่ 8 มกราคม พ.ศ.2562

ชื่อ-สกุล.. รหัสนักศึกษา............................. ลำดับที่............

1. กำหนดให้ f นิยามดังภาพ ให้หาลิมิตต่อไปนี้ โดยเติมคำตอบลงในช่องว่าง



(a) $\lim\limits_{x\to -1^-} f(x) =$ ...-1...

(b) $\lim\limits_{x\to -1^+} f(x) =$ ...$-\infty$...

(c) $\lim\limits_{x\to 0^-} f(x) =$ ...0...

(d) $\lim\limits_{x\to 0^+} f(x) =$ ...0...

(e) $\lim\limits_{x\to 1^-} f(x) =$ ...1...

(f) $\lim\limits_{x\to 1^+} f(x) =$ ...$+\infty$...

(g) The vertical asymtotes of the graph of f (เส้นกำกับแนวยืนของกราฟ f) is ...$x = -1, 1$...

2. ให้หาลิมิตต่อไปนี้

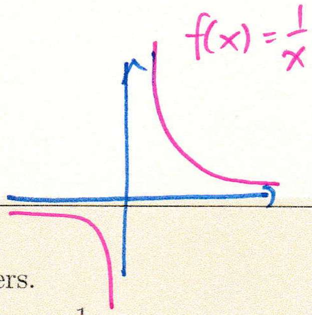
(a) $\lim\limits_{x\to 1^-} \dfrac{x-3}{(x-1)(x-2)^2} =$ ....

(b) $\lim\limits_{x\to 1^+} \dfrac{x-3}{(x-1)(x-2)^2} =$ ....

(c) $\lim\limits_{x\to 1} \dfrac{x-1}{(x-3)(x-2)^2} =$ ....

(d) $\lim\limits_{x\to 1} \dfrac{x+1}{(x-3)(x-2)^2} =$ ....

1.2 Computing Limits



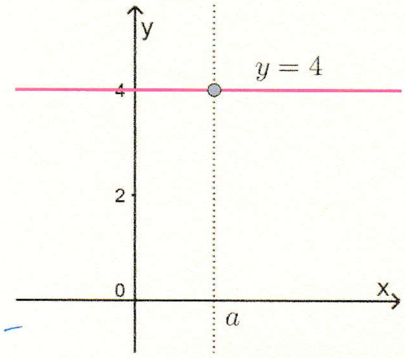
1.2.1 Some Basic Limits

Theorem 1.1 Let a and k be real numbers.

(a) $\lim\limits_{x \to a} k = k$ (b) $\lim\limits_{x \to a} x = a$ (c) $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$ (d) $\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$

Example 1.7 If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $\lim\limits_{x \to a} k = k$ for all value of a. For example,

Solution.



$\lim\limits_{x \to -5} 4 =$ 4

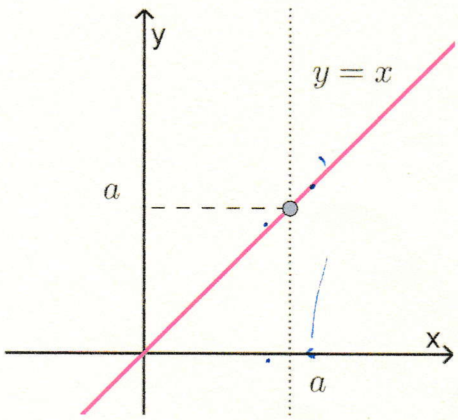
$\lim\limits_{x \to 0} 4 =$ 4

$\lim\limits_{x \to 15} 4 =$ 4

Figure 1.9: Graph of $f(x) = 4$

Example 1.8 If $f(x) = x$, then the values of $f(x)$ always equals to x varies, which explains why $\lim\limits_{x \to a} x = a$ for all value of a. For example,

Solution.



$\lim\limits_{x \to 0} x =$ 0

$\lim\limits_{x \to 3} x =$ 3

$\lim\limits_{x \to -100} x =$ -100

Figure 1.10: Graph of $f(x) = x$

THEOREM 1.2 Let a be a real number, and suppose that

$$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} g(x) = L_2.$$ (ถ้าหาค่าได้)

That is, the limits exist and have values L_1 and L_2, respectively, Then:

(a) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(b) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(c) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, provided $L_2 \neq 0$

(e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if n is even.

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

1.2.2 Limits of Polynomials and Rational Functions as $x \to a$

Example 1.9 Find $\lim_{x\to 2}(x^3 - 3x + 5)$.

Solution.

$$\lim_{x\to 2}(x^3-3x+5) = \lim_{x\to 2} x^3 - \lim_{x\to 2} 3x + \lim_{x\to 2} 5$$
$$= (2)^3 - 3(2) + 5$$
$$= 8 - 6 + 5$$
$$= 2 + 5$$
$$= 7 \quad \#$$

⊛ $\lim_{x\to 2}(x^3 - 3x + 5) = (2)^3 - 3(2) + 5 = 7$

THEOREM 1.3 For any polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Ex 1.1

$$\lim_{x \to a} f(x) = 2, \quad \lim_{x \to a} g(x) = -4, \quad \lim_{x \to a} h(x) = 0$$

1. $$\lim_{x \to a} [f(x) + 2g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} 2g(x)$$
$$= \lim_{x \to a} f(x) + 2\lim_{x \to a} g(x)$$
$$= 2 + 2(-4) = -6$$

2. $$\lim_{x \to a} [f(x) \cdot g(x)] = -8$$

3. $$\lim_{x \to a} [h(x) \div 3g(x) + 2] = 0 - 3(-4) + 2 = -10$$

4. $$\lim_{x \to a} [g(x)]^2 = [\lim_{x \to a} g(x)]^2 = 16$$

5. $$\lim_{x \to a} \sqrt[3]{6 + f(x)} = \sqrt[3]{\lim_{x \to a} (6 + f(x))} = \sqrt[3]{6+2} = \sqrt[3]{8} = 2$$

6. $$\lim_{x \to a} \frac{2}{g(x)} = \frac{\lim_{x \to a} 2}{\lim_{x \to a} g(x)} = \frac{2}{-4} = -\frac{1}{2}$$