ทบทวน

f เป็นฟังก์ชัน

อนุพันธ์ของ f ที่จุด x ใดๆ

$$\frac{df}{dx} = \frac{dy}{dx} = \boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

อนุพันธ์ของ f ที่จุด x_0

$$\boxed{f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}$$

ฟังก์ชัน f หาอนุพันธ์ได้ (differentiable) ที่จุด x_0

ถ้า $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ มีค่า.

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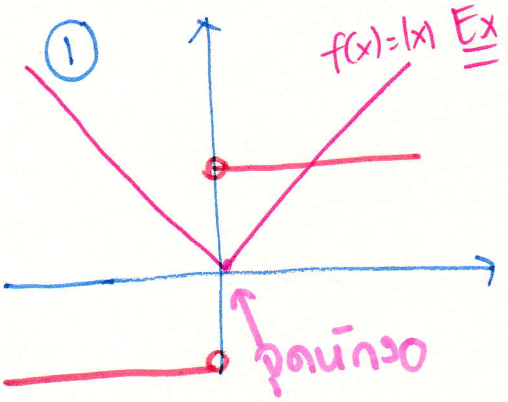
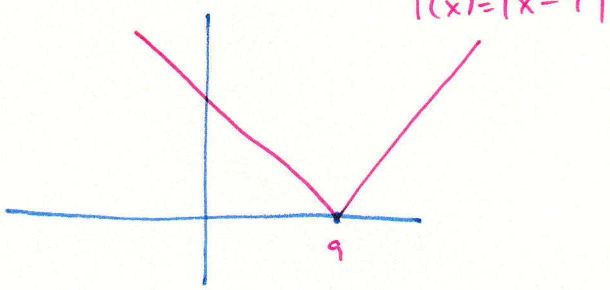
(1) **Ex** $f(x) = |x|$ หาอนุพันธ์ไม่ได้ที่ $x=0$



↓

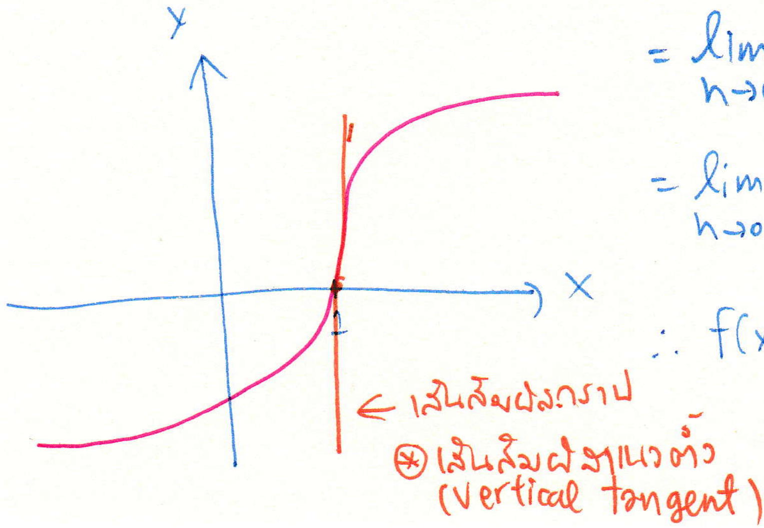
$$f(x) = |x-9|$$



(2) $f(x) = (x-1)^{\frac{1}{3}}$ ดูว่าหาอนุพันธ์ได้ที่ $x=1$ หรือไม่?

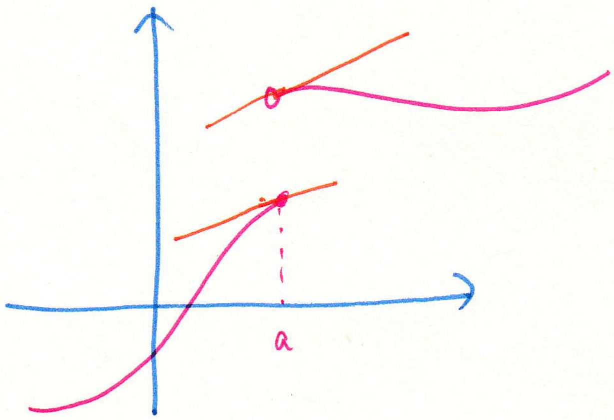
พิจารณา $\boxed{f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}}$ มีค่าไหม?

ดังนั้น
$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h-1)^{\frac{1}{3}} - \cancel{(1-1)^{\frac{1}{3}}}^{\,0}}{h}$$
$$= \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$$



$\therefore f(x) = (x-1)^{\frac{1}{3}}$ หาอนุพันธ์ไม่ได้ที่ $x=1$

③ จุดที่ฟังก์ชันไม่ต่อเนื่อง ก็จะหาอนุพันธ์ไม่ได้



$\boxed{\text{Theorem 2.1}}$ ถ้า f หาอนุพันธ์ได้ (ที่ x_0) แล้ว f จะต่อเนื่อง (ที่ x_0)

บทกลับ ถ้า f ต่อเนื่องที่ x_0 แล้ว f หาอนุพันธ์ได้ที่ x_0

ไม่จริงเสมอไป $f(x) = |x|$

เป็นฟังก์ชันต่อเนื่องที่ $x=0$

แต่หาอนุพันธ์ไม่ได้ที่ $x=0$

$\boxed{y = f(x)}$

อนุพันธ์ที่ x ใดๆ →	อนุพันธ์ที่ x_0
• $f'(x)$	• $f'(x_0)$
• $\frac{dy}{dx}$	• $\left.\frac{dy}{dx}\right\vert_{x=x_0}$
• $\frac{d}{dx}[f(x)]$	• $\left.\frac{d}{dx}[f(x)]\right\vert_{x=x_0}$
• $D_x[f(x)]$	• $y'(x_0)$
• y'	
• $y'(x)$	

การหาอนุพันธ์โดยใช้สูตร

2.3 Basic Differentiation Formulas

RECAP Henceforth, we will use the following notation for the derivative of a function $y = f(x)$.

- $f'(x)$
- $\frac{d}{dx}[f(x)]$
- $\frac{dy}{dx}$
- y'

Derivative of a constant

THEOREM 2.2 (THE CONSTANT RULE) Let c be a constant and $f(x) = c$ be a constant function, then

$$\frac{d}{dx}[c] = 0$$

Proof หา $f'(x)$ เมื่อ $f(x) = c$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{c - c}{h} = 0$$

Example 2.9

$\frac{d}{dx}[5] = \ldots 0 \ldots$ $\quad \frac{d}{dx}[-7] = \ldots 0 \ldots$ $\quad \frac{d}{dx}[5^2] = \ldots 0 \ldots$

$\frac{d}{dx}[\pi] = \ldots 0 \ldots$ $\quad \frac{d}{dx}[\sqrt{2}] = \ldots 0 \ldots$ $\quad \frac{d}{dx}[\pi^2] = \ldots 0 \ldots$

$\frac{d}{dx}[e] = 0 \quad \frac{d}{dx}[e^{\pi}] = 0 \quad \frac{d}{dx}[\pi^e] = 0$

$\boxed{\frac{d}{dx}[e^x] = ?}$

Derivatives of power functions

$\frac{d}{dx}(x^x) = \cancel{x\,x^{x-1}}$

Theorem 2.3 (The Power Rule) If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

x เป็นตัวแปร n เป็นตัวเลข

Example 2.10

$\frac{d}{dx}[x^5] = 5x^4$ $\qquad \frac{d}{dx}[x] = 1$ $\qquad \frac{d}{dx}[x^{12}] = 12x^{11}$

$\frac{d}{dx}[x^{-5}] = -5x^{-6}$ $\qquad \frac{d}{dx}[\frac{1}{x^5}] = \frac{d}{dx}(x^{-5}) = -5x^{-6} = \frac{-5}{x^6}$ $\qquad \frac{d}{dx}[x^{\sqrt{2}}] = \sqrt{2}\,x^{\sqrt{2}-1}$

$\frac{d}{dx}[x^{\frac{2}{5}}] = \frac{2}{5}x^{\frac{2}{5}-1} = \frac{2}{5}x^{-\frac{3}{5}}$ $\qquad \frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\qquad \frac{d}{dx}[\frac{1}{\sqrt{x}}] = \frac{d}{dx}(x^{-\frac{1}{2}}) = (-\frac{1}{2})x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$

$\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$

Derivative of a constant times a fucntion

Theorem 2.4 (The Constant Multiple Rule) If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example 2.11 $\frac{d}{dx}[4x^8] = 4\left(\frac{d}{dx}(x^8)\right) = 4(8x^7) = 32x^7$

$\frac{d}{dx}[-x^{12}] = -12x^{11}$

$\frac{d}{dx}[\frac{x}{\pi}] = \frac{1}{\pi}$

Derivatives of sums and diffences

THEOREM 2.5 (THE SUM AND DIFFERENCE RULES) If f and g are differentiable at x, then so are $f+g$ and $f-g$ and

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$\boxed{\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}}$$

Example 2.12 $\frac{d}{dx}[x^8+x^2] = \frac{d}{dx}(x^8) + \frac{d}{dx}(x^2) = 8x^7 + 2x$

$\frac{d}{dx}[6x^{11}-9] = 66x^{10} + 0$

$\frac{d}{dx}[\frac{1}{x} - x^{-9}] = \frac{d}{dx}[x^{-1} - x^{-9}] = -x^{-2} + 9x^{-10} = -\frac{1}{x^2} + \frac{9}{x^{10}}$

$\frac{d}{dx}[\frac{\sqrt{x}-3x}{\sqrt{x}}] = \frac{d}{dx}[\frac{\sqrt{x}}{\sqrt{x}} - \frac{3x}{\sqrt{x}}] = \frac{d}{dx}[1 - 3\sqrt{x}] = 0 - \frac{3}{2\sqrt{x}} = \frac{-3}{2\sqrt{x}}$

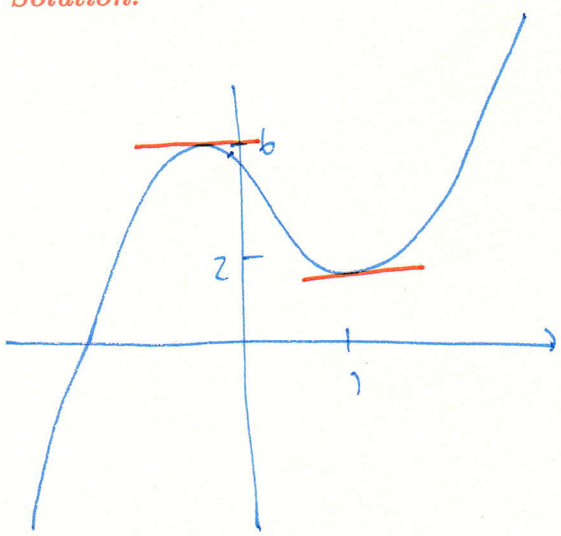
Example 2.13 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

เส้นสัมผัสแนวนอน

หาจุดที่ทำให้ความชัน เป็น 0.

Solution.

$\boxed{f(x) = x^3 - 3x + 4}$



หาจุด x ที่ทำให้ $\boxed{f'(x) = 0}$

$f'(x) = 3x^2 - 3$

Set $f'(x) = 0$

$\therefore 3x^2 - 3 = 0$

$x^2 - 1 = 0$

$\boxed{x = \pm 1}$

ที่ $x = \pm 1$ ทำให้ f.เกิดความชันเท่ากับ 0

ถ้า $x = 1$, $f(1) = 2$
$x = -1$, $f(-1) = 6$ } จุดที่มีความชันเป็น 0 คือ จุด $(1,2), (-1,6)$.

2.3.1 Higher derivatives อนุพันธ์อันดับสูง

The derivative f' of a function f is itself a function and hence may have its own derivative. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as we have differentiability, we can continue the process of differentiating to obtain the third, the fourth, the fifth, and even higher derivatives of f. These successive derivatives are denoted by

the first derivative	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$
the second derivative	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$
the third derivative	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$
the fourth derivative	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A general nth order derivative	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$

Example 2.14 If $f(x) = 6x^4 - 5x^3 + x^2 - 7x + 8$,then

$f'(x) = 24x^3 - 15x^2 + 2x - 7$

$f''(x) = 72x^2 - 30x + 2$

$f'''(x) = 144x - 30$

$f^{(4)}(x) = 144$

$f^{(5)}(x) = 0$

$f^{(500)}(x) = 0$

Example 2.15 Let $y = \frac{x+1}{x}$. Find $y^{(111)}$.

Solution. $y = 1 + \frac{1}{x} = 1 + x^{-1}$

$$\boxed{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n = n!}$$

$$y' = (-1)x^{-2}$$

$$y'' = (-1)(-2)x^{-3} \longrightarrow (-1)^2(1\cdot 2)x^{-3} = (-1)^2 \cdot 2!\,x^{-3}$$

$$y''' = (-1)(-2)(-3)x^{-4} \longrightarrow (-1)^3(1\cdot 2\cdot 3)x^{-4} = (-1)^3 \cdot 3!\,x^{-4}$$

$$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5} \longrightarrow (-1)^4(1\cdot 2\cdot 3\cdot 4)x^{-5} = (-1)^4 \cdot 4!\,x^{-5}$$

$$\vdots$$

$$y^{(111)} = (-1)(-2)(-3)\cdot\ldots\cdot(-111)x^{-112} = (-1)^{111}\,111!\,x^{-112}$$

$$y^{(2019)} = (-1)^{2019}\,2019!\,x^{-2020}$$

$$\downarrow$$

$$y^{(n)} = (-1)^n n!\,x^{-n-1} = \frac{(-1)^n n!}{x^{n+1}}$$

2.4 The Product and Quotient Rules

2.4.1 Derivative of a Product

THEOREM 2.6 (THE PRODUCT RULE) If f and g are differentiable at x, then so is the product $f \cdot g$, and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

(นล)' = นล' + ลน'

Example 2.16 1. Let $y = (4x^2 - 1)(7x^3 + x)$. Find dy/dx.

Solution. คูณกระจาย $y = (4x^2-1)(7x^3+x) = 28x^5 - 3x^3 - x$

$$\frac{dy}{dx} = 140x^4 - 9x^2 - 1$$

ใช้ diff ผลคูณ $\frac{dy}{dx} = (4x^2-1)\frac{d}{dx}(7x^3+x) + (7x^3+x)\frac{d}{dx}(4x^2-1)$

$$= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) \quad \#$$

$$= 140x^4 - 9x^2 - 1$$

2. Let $f(x) = (3x + 1)(2x^2 + 5)$. Find $f'(0)$.

Solution. $\frac{dy}{dx} = f'(x) = (3x+1)\frac{d}{dx}(2x^2+5) + (2x^2+5)\frac{d}{dx}(3x+1)$

$$= (3x+1)(4x) + (2x^2+5)(3)$$

$$f'(0) = \overbrace{(3(0)+1)(4(0))}^{0} + (2(0)^2+5)(3)$$

$$= 15 \quad \#$$

2.4.2 Derivative of a Quotient

Theorem 2.7 (The Quotient Rule) If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\left(\frac{ข}{ล}\right)' = \frac{ลข' - ขล'}{ล^2}$$

Example 2.17 Find dy/dx for y.

1. $y = \dfrac{\boxed{x^2 - 1}}{\boxed{x^4 + 1}}$

Solution.

$$y' = \frac{dy}{dx} = \frac{(x^4+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$$

$$= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2} \quad \#$$

จัดรูป $\vdots$

$$= -\frac{2x(x^4-2x^2-1)}{(x^4+1)^2}$$

2. $y = \dfrac{\boxed{2x + 1}}{\boxed{\sqrt{x}}}$ $\quad \left(y = 2\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

Solution.

$$\frac{dy}{dx} = \frac{\sqrt{x}\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} - (2x+1)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{2\sqrt{x}}{x} - \frac{2x+1}{x}\cdot\frac{1}{2\sqrt{x}} \quad \#$$

จัดรูป $\vdots$

$$= \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$

Example 2.3 Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $x_0 = 1$, $x_0 = 4$, and x_0=9.

Solution.

Rectilinear Motions

$f(t)$: ตำแหน่งของวัตถุเคลื่อนที่ ณ เวลา t

อัตราเร็วเฉลี่ย ~~ความเร็ว~~

The position coordinate of a particle in rectilinear motion at time t is $s = f(t)$. The ***average velocity*** of the particle over a time interval $[t_0, t_0 + h]$ for $h > 0$ is

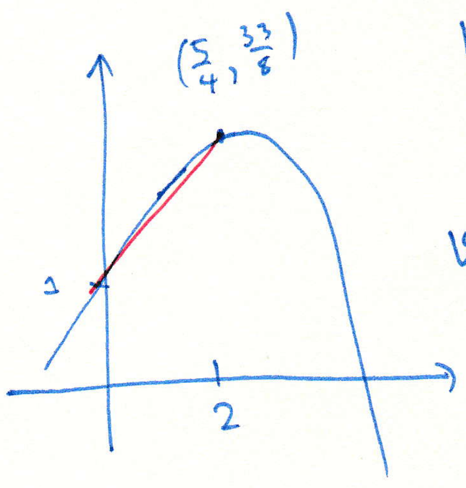
การกระจัด (displacement)

$$v_{ave} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h}.$$

$s = vt \rightarrow \boxed{v = \frac{s}{t}}$

Example 2.4 Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.

Solution. ความเร็วเฉลี่ย บน $[0,2]$



$$v_{ave[0,2]} = \frac{f(2)-f(0)}{2-0} = \frac{(1+5(2)-2(2)^2)-(1+5\cdot0-2\cdot0)}{2}$$

$$= \frac{(1+10-8)-1}{2} = 1 \text{ m/s.}$$

$$v_{ave[2,3]} = \frac{f(3)-f(2)}{3-2} = \frac{f(2+1)-f(2)}{1}$$

$$= \frac{[1+5\cdot3-2(5)^2]-[1+5\cdot2-2\cdot5^2]}{1}$$

$$= -5 \text{ m/s.}$$

ความเร็วณ ขณะใด ขณะหนึ่ง

Taking the limit of v_{ave} as $h \to 0$, we get instantaneous velocity v_{inst} of the particle at time t_0.

$$\boxed{v_{inst} = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}}. = f'(t_0)$$

Example 2.5 Consider particle in Example 2.4, where

$$s = f(t) = 1 + 5t - 2t^2.$$

The position of the particle at time $t = 2$ s is $s = 3$ m. Find the instantaneous velocity of the particle at time $t = 2$ s.

Solution.

ความเร็ว ณ $t = 2$

$s = f(t) = 1 + 5t - 2t^2$

ความเร็ว ณ จุด $t = 2$: $f'(2)$

$f(t) = 1 + 5t - 2t^2$

$f'(t) = 5 - 4t$

$\therefore f'(2) = 5 - 8 = -3$ m/s.

สมมติว่า มีปริมาณระบุมาให้เป็นฟังก์ชัน ณ เวลา t.

- ความเปลี่ยนแปลงเฉลี่ย ช่วง $[a,b]$ $= \dfrac{f(b) - f(a)}{b - a}$
- ความเปลี่ยนแปลง ณ เวลาใดเวลาหนึ่ง (ณ เวลา t_0) $= \lim_{h \to 0} \dfrac{f(t_0 + h) - f(t_0)}{h} = f'(t_0)$.