ทบทวน

Ex $$\lim_{x\to+\infty} x^5+3x^4-x^3+x^2-1 = \lim_{x\to\infty} x^5\left(1-\frac{3}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^5}\right)=+\infty$$

Ex $$\lim_{x\to\infty} \frac{\sqrt{x^4+x}}{x^3-x} = \lim_{x\to\infty}\frac{\sqrt{x^4\left(1+\frac{1}{x^3}\right)}}{x(x^2-1)}$$ $\Rightarrow$ $\boxed{\sqrt{x^2}=|x|=\begin{cases}x; & x\geq 0\\ -x; & x<0\end{cases}}$

$$=\lim_{x\to+\infty}\frac{|x^2|\sqrt{1+\frac{1}{x^3}}}{x(x^2-1)}$$

$|x^2|=\begin{cases}x^2; & x^2\geq 0\\ \cancel{-(x^2); \; x^2<0}\end{cases}$

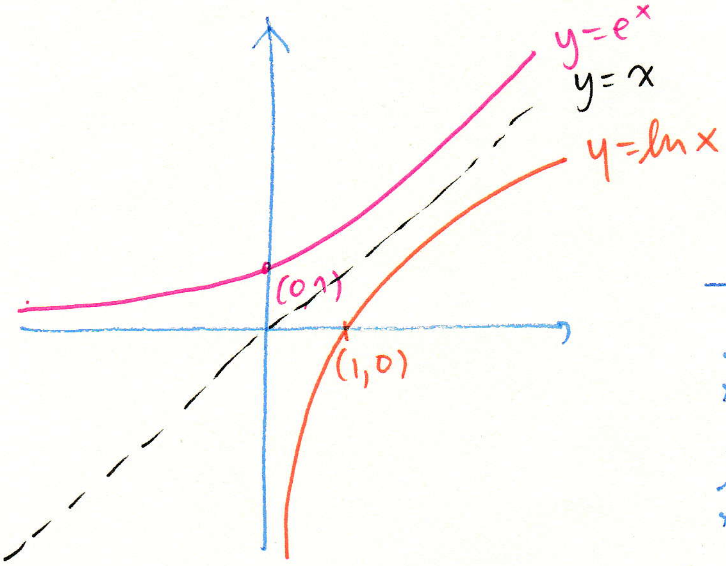
$$=\lim_{x\to+\infty}\frac{x^{\cancel{2}}\sqrt{1+\frac{1}{x^3}}}{\cancel{x}(x^2-1)}$$

$$=\lim_{x\to+\infty}\frac{x\sqrt{1+\frac{1}{x}}}{x^2-1}$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{x^2}\sqrt{1+\frac{1}{x}}}{\frac{\cancel{x^2}}{\cancel{x^2}}-\frac{1}{x^2}}$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{x^2}\sqrt{1+\cancel{\frac{1}{x}}^{\,0}}}{1-\cancel{\frac{1}{x^2}}^{\,0}}$$

$$=0. \qquad \#$$



$$\lim_{x\to+\infty} e^x = +\infty$$

$$\lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to+\infty} \ln x = +\infty$$

$$\lim_{x\to 0^+} \ln x = -\infty$$

3.6 Indeterminate Forms

3.6.1 L'Hospital's Rule

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0, 0 \cdot \infty$

THEOREM 3.4 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

เชคให้มั่นใจว่า เป็น $\boxed{\frac{0}{0}}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the theorem also works for one-sided limits.

Example 3.25 Find $\lim_{x\to 0} \frac{3x - \sin x}{x}$.

$\left.\begin{array}{l} \lim_{x\to 0} 3x - \sin x = 0 \\ \lim_{x\to 0} x = 0 \end{array}\right\}$ $\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(3x - \sin x)}{\frac{d}{dx}(x)}$

$$= \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3-1}{1} = 2$$

THEOREM 3.5 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

indeterminate $\frac{\infty}{\infty}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the theorem also works for one-sided limits.

Example 3.26 Find $\lim_{x\to\infty} \frac{2x + 3}{5x + 7}$. $\rightarrow \lim_{x\to\infty} \frac{\frac{2x}{x} + \frac{3}{x}}{\frac{5x}{x} + \frac{7}{x}} = \lim_{x\to\infty} \frac{2 + \frac{3}{x}^{\,0}}{5 + \frac{7}{x}^{\,0}} = \frac{2}{5}$

$\left.\begin{array}{l} \lim_{x\to\infty} 2x + 3 = +\infty \\ \lim_{x\to+\infty} 5x + 7 = +\infty \end{array}\right\}$ $\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2x+3)}{\frac{d}{dx}(5x+7)} = \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5}$

$\underline{\underline{Ex}}$ $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \frac{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}{\cancel{\sqrt{x}-3}}$

$= \lim_{x\to 9} \sqrt{x}+3 = 6$

$\underline{Ex}$ $\lim_{x\to 0} \frac{3x - \sin x}{x}$ (ทำแล้ว)

$\underline{Ex}$ $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x\to 2} (x+2) = 4$

ใช้ L'Höpital

$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{2x}{1} = 2(2) = 4$

$\underline{Ex}$ $\lim_{x\to 0} \frac{x+6}{x+2}$

ใช้ไม่ได้ เพราะไม่เป็น $\frac{0}{0}$

ใช้ L'Höpital $\cancel{\lim_{x\to 0} \frac{x+6}{x+2} = \lim_{x\to 0} \frac{\frac{d}{dx}(x+6)}{\frac{d}{dx}(x+2)} = \lim_{x\to 0} \frac{1}{1} = 1}$

$\lim_{x\to 0} \frac{x+6}{x+2} = \frac{6}{2} = 3$

$\underline{Ex}$ $\lim_{x\to 0^+} \frac{\tan x}{x^2}$

ใช้ L'Höpital : $\lim_{x\to 0^+} \frac{\tan x}{x^2} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\tan x)}{\frac{d}{dx}(x^2)} = \lim_{x\to 0^+} \frac{\sec^2 x}{2x} = +\infty$

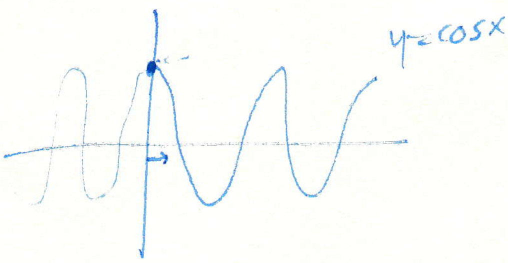
1

0

$\cancel{= \frac{1}{0^+}}$

$\underline{Ex}$ $\lim_{x\to 0^+} \frac{2+x^2}{1-\cos x} = +\infty$

0



Ex ให้ f และ g เป็นฟังก์ชันซึ่ง $\lim_{x\to 1} f(x) = -\infty$, $\lim_{x\to 1} g(x) = +\infty$

$\lim_{x\to 1} f'(x) = 4$, $\lim_{x\to 1} g'(x) = 2$.

$\boxed{1}$ $\lim_{x\to 1} \frac{111}{g(x)} = 0$

$\boxed{2}$ $\lim_{x\to 1} \frac{g(x)}{f(x)} = \lim_{x\to 1} \frac{g'(x)}{f'(x)} = \frac{2}{4} = \frac{1}{4}$

แปลงร่าง

$$\infty \cdot 0 \longrightarrow \underbrace{\frac{0}{0} \,/\, \frac{\infty}{\infty}}_{\text{ใช้ L'Hôpital ได้}} \qquad \Bigg| \qquad A \cdot B = \frac{A}{\frac{1}{B}} \,\Big/\!\!\Big/\, \frac{B}{\frac{1}{A}}$$

Other Indeterminate forms

Solving other indeterminate forms require us to change the limit so that we can use L'hospital's rule.

Example 3.27 Find $\lim\limits_{x\to\infty} x\tan\frac{1}{x}$ $\quad +\infty \cdot 0$

$x \to \infty,\ \frac{1}{x} \to 0 \Rightarrow \tan(\frac{1}{x}) \to 0$

เป็นรูปอินดีเทอร์มิเนตแบบ $\infty \cdot 0$

$$\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x})\cancel{\frac{d}{dx}(\frac{1}{x})}}{\cancel{\frac{d}{dx}(\frac{1}{x})}} = \lim_{x\to\infty} \sec^2(\frac{1}{x})$$

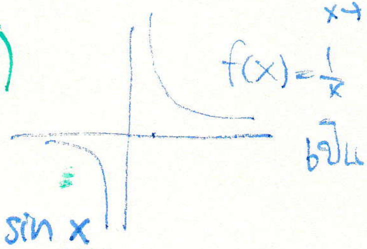
เมื่อ $x \to \infty$ ได้ $\frac{1}{x} \to 0$

$\therefore \sec^2(\frac{1}{x}) \to 1$

$\therefore \lim\limits_{x\to\infty} x\tan(\frac{1}{x}) = 1$ #

$$\lim_{x\to\infty} \frac{x}{\frac{1}{\tan(\frac{1}{x})}} = \lim_{x\to\infty} \frac{x}{\cot(\frac{1}{x})}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{1}{-\operatorname{cosec}^2(\frac{1}{x})\frac{d}{dx}(\frac{1}{x})}$$



Example 3.28 Find $\lim\limits_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $\quad \infty - \infty$

เป็นรูปอินดีเทอร์มิเนตแบบ $\infty - \infty$

$$\therefore \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x - \sin x}{x\sin x}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(x\sin x)} = \lim_{x\to 0^+} \frac{1 - \cos x}{x\cos x + \sin x}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to 0^+} \frac{\sin x}{(x(-\sin x) + \cos x + \cos x}$$

$$= 0 \quad \#$$

In case the limit of the form $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^\infty$, apply logarithm to the limit.

Example 3.29 Find $\lim\limits_{x\to 0^+} (\sin x)^x$. เป็นรูปอินดีเทอร์มิเนตแบบ 0^0

สมมติให้ $y = (\sin x)^x$

Take ln $\quad \ln y = \ln(\sin x)^x$

$\ln y = x\ln(\sin x) \quad (0 \cdot \infty)$

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \boxed{x\ln(\sin x)}$

พิจารณา $\quad \lim\limits_{x\to 0^+} x\ln(\sin x) = \lim\limits_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}}$

$$\overset{\text{L'Höpital}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln(\sin x))}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\frac{d}{dx}(\sin x)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\cot x}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\tan x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -\frac{x^2}{\tan x}$$

$$\overset{\text{L'Höpital}}{=} \lim_{x\to 0^+} -\frac{2x}{\sec^2 x} = 0$$

เคยพูดว่า $\lim_{x\to 0^+} \ln y = \boxed{\lim_{x\to 0^+} x\ln(\sin x)} = 0$

$$\therefore \lim_{x\to 0^+} \ln y = 0$$

พิจารณาว่า $\ln y$ ต่อเนื่องบนช่วง $(0,\infty)$

อยากได้ $y \to e^{\ln y} = y$

ฟังก์ชัน $f(x)=e^x$ ต่อเนื่องทุกจุดบน x.

และ $\boxed{\lim_{x\to 0^+} \ln y = 0}$

โดย Thm 1.7. $\lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^0 = 1$

$$\lim_{x\to 0^+} \ln y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$$

$\log_{\text{ฐาน}} x = \text{ตัวเลข} \iff x = \text{ฐาน}^{\text{ตัวเลข}}$ → $\lim_{x\to 0^+} \log_e y = 0 \Rightarrow \lim_{x\to 0^+} y = e^0 = 1$

Failures of L'Hospital's Rule

Sometimes using L'hospital's Rule will not solve the problem as in following example.

Example 3.30 Find $\displaystyle\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$.

Failed Solution First, let find out what happens if we use L'Hospital's rule.

Notice that $\displaystyle\lim_{x\to(\pi/2)^-} \sec x = \infty = \lim_{x\to(\pi/2)^-} \tan x$. Therefore, it is possible to use L'Hoslpital's rule.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{\frac{d}{dx}\sec x}{\frac{d}{dx}\tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}.$$

Using L'Hospital's rule again, we will have

$$\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}.$$

So we return to the problem at the beginning and nothing is solved!

Good Solution *Try simplifying the expression first.*

$$\frac{\sec x}{\tan x} = \frac{1/\cos x}{\sin x/\cos x} = \frac{1}{\sin x}.$$

Therefore,

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{1}{\sin x} = 1.$$