

$$\int \sqrt{a^2 - x^2} dx$$

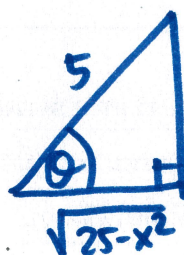
$$\begin{cases} \sqrt{a^2 - u^2} \Rightarrow u = a \sin \theta \\ \sqrt{a^2 + u^2} \Rightarrow u = a \tan \theta \\ \sqrt{u^2 - a^2} \Rightarrow u = a \sec \theta \end{cases}$$

Ex $\int \frac{\sqrt{25-x^2}}{100x^2} dx$

माना कि

$$x = 5 \sin \theta$$

$$dx = 5 \cos \theta d\theta$$



$$\sin \theta = \frac{x}{5} \Rightarrow \theta = \arcsin \left(\frac{x}{5} \right)$$

$$\cos \theta = \frac{\sqrt{25-x^2}}{5}$$

$$\therefore \int \frac{\sqrt{25-x^2}}{100x^2} dx = \int \frac{\sqrt{25-25\sin^2\theta}}{25 \cdot 100 \sin^2\theta} \cdot 5 \cos \theta d\theta$$

$$= \int \frac{5 \sqrt{(1-\sin^2\theta)}}{25 \cdot 100 \sin^2\theta} \cdot 5 \cos \theta d\theta$$

$$= \frac{1}{100} \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \frac{1}{100} \int \cot^2 \theta d\theta$$

$$= \frac{1}{100} \int \cot \theta \cdot \cos \theta \cdot \operatorname{cosec} \theta d\theta$$

$$= \frac{1}{100} \int (\operatorname{cosec}^2 \theta - 1) d\theta$$

$$= \frac{1}{100} [-\cot \theta - \theta] + C$$

$$= \frac{1}{100} \left[-\frac{\sqrt{25-x^2}}{x} - \arcsin \left(\frac{x}{5} \right) \right] + C$$

$$= \frac{1}{100} \left[-\frac{\sqrt{25-x^2}}{x} - \arcsin \left(\frac{x}{5} \right) \right] + C$$

Ex 7.11

$$\int \frac{dx}{x^2 \sqrt{4x^2 - 3}} \rightarrow a = \sqrt{3}$$

$u = 2x$

สมมติให้ $2x = \sqrt{3} \sec \theta \Rightarrow x = \frac{\sqrt{3}}{2} \sec \theta$

$$dx = \frac{\sqrt{3}}{2} \sec \theta \tan \theta d\theta$$

$$\int \frac{dx}{x^2 \sqrt{4x^2 - 3}} = \int \frac{\frac{\sqrt{3}}{2} \sec \theta \tan \theta d\theta}{3 \sec^2 \theta \sqrt{3 \sec^2 \theta - 3}}$$

$$= \frac{1}{2} \int \frac{\cancel{\sqrt{3}} \sec \theta \tan \theta d\theta}{3 \sec^2 \theta \cdot \sqrt{3} (\tan \theta)}$$

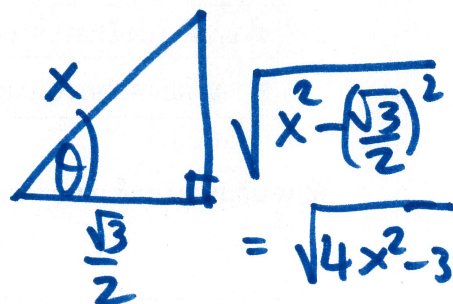
$\frac{1}{42}$

$$= \frac{2}{3} \int \frac{1}{\sec \theta} d\theta$$

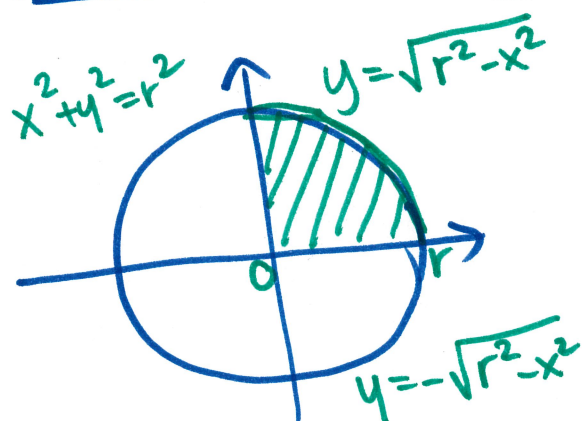
$$= \frac{2}{3} \int \cos \theta d\theta$$

$$= \frac{2}{3} [\sin \theta] + C$$

$$= \frac{2}{3} \left[\frac{\sqrt{4x^2 - 3}}{x} \right] + C \quad \#$$



การหาคำตอบ สำหรับ w.h. ของวงกลมรัศมี r คือ πr^2



$$A = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$A = \pi r^2$$

Ex

$$\int \frac{e^x}{\sqrt{e^{2x} + e^x + 1}} dx$$

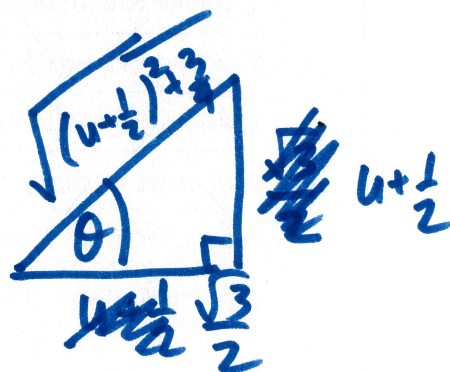
bu $u = e^x$; $du = e^x dx$

$$\begin{aligned} \therefore \int \frac{e^x dx}{\sqrt{e^{2x} + e^x + 1}} &= \int \frac{du}{\sqrt{u^2 + u + 1}} \\ &= \int \frac{du}{\sqrt{(u^2 + 2 \cdot u \cdot \frac{1}{2} + (\frac{1}{2})^2) - \frac{1}{2^2} + 1}} \\ &= \int \frac{du}{\sqrt{(u + \frac{1}{2})^2 + \frac{3}{4}}} \quad \text{--- } \otimes \end{aligned}$$

bu $u + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$

$\therefore du = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$

$$\begin{aligned} \otimes \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\sqrt{\frac{3}{4} \tan^2 \theta + \frac{3}{4}}} &= \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\frac{\sqrt{3}}{2} \sec \theta} = \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \end{aligned}$$



$$= \ln \left| \frac{\sqrt{(u + \frac{1}{2})^2 + \frac{3}{4}}}{\frac{\sqrt{3}}{2}} + \frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right| + C$$

$$= \ln \left| \frac{2\sqrt{(e^x + \frac{1}{2})^2 + \frac{3}{4}}}{\sqrt{3}} + \frac{2(e^x + \frac{1}{2})}{\sqrt{3}} \right| + C$$

$$= \ln \left| \frac{2\sqrt{e^{2x} + e^x + 1}}{\sqrt{3}} + \frac{2e^x + 1}{\sqrt{3}} \right| + C$$

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#

ឧទាហរណ៍លេខ ៤

$$f(x) = \frac{p(x)}{q(x)} \quad ; \quad \begin{array}{l} p(x) \text{ ជា ឈុត} \\ q(x) \text{ ជា ឈុត} \end{array}$$

$f(x)$ ជា អនុគមន៍លីនេអ៊ែរ (rational function)

$$\underline{\text{Ex}} \quad \int \frac{x^2 - x + 1}{x^3 + x - 1} dx \quad ? ! ?$$

ឧទាហរណ៍លេខ ៤ ក៏: ពិត ចិត្ត ក្នុង ក្រុង ក្រុង

$$f(x) = \frac{p(x)}{q(x)}$$

ឧទាហរណ៍លេខ ៤
partial fraction

$$\deg(p(x)) < \deg(q(x))$$

$$\underline{\text{Ex}} \quad \int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$$

$$\begin{array}{r} + 1 \\ x^2 + x - 2 \overline{) 3x^4 + 3x^3 - 5x^2 + x - 1} \\ \underline{+ 3x^4 + 3x^3 - 6x^2} \\ x^2 + x - 1 \\ \underline{x^2 + x - 2} \\ 1 \end{array}$$

$$3x^4 + 3x^3 - 5x^2 + x - 1 = (x^2 + x - 2)(\underline{\underline{3x^2 + 1}}) + 1$$

$$\therefore \int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx = \int \left[(3x^2 + 1) + \left(\frac{1}{x^2 + x - 2} \right) \right] dx$$

Ansatz von $\frac{1}{x^2 + x - 2} = \text{Heart} + \text{Flower}$

$$f(x) = \frac{p(x)}{q(x)} ; \deg(p(x)) < \deg(q(x))$$

წილის ფუნქცია (proper rational fⁿ)

$$\textcircled{1} \frac{1}{(x-a_1)(x-a_2)} = \frac{A}{x-a_1} + \frac{B}{x-a_2}$$

$$A = ? , B = ?$$

$$\textcircled{2} \frac{1}{\boxed{(x-a_1)^2} \boxed{(x-a_2)^3} \boxed{(x-a_3)}} = \boxed{\frac{A}{x-a_1} + \frac{B}{(x-a_1)^2}} + \boxed{\frac{C}{x-a_2} + \frac{D}{(x-a_2)^2} + \frac{E}{(x-a_2)^3}} + \boxed{\frac{F}{x-a_3}}$$

Ex

$$\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x+2)(x-1)}$$

we know $\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$$\boxed{1 = A(x-1) + B(x+2)}$$

$$1 = \underline{Ax} - A + \underline{Bx} + 2B$$

$$\underline{0x} + \underline{1} = \underline{(A+B)x} + \underline{(-A+2B)}$$

by equating

$$A+B=0 \quad \text{---(1)}$$

$$-A+2B=1 \quad \text{---(2)}$$

$$\textcircled{1} + \textcircled{2}; \quad 3B=1 \Rightarrow B=\frac{1}{3}$$

$$\therefore A = -\frac{1}{3}$$

$$\therefore \int \frac{1}{(x+2)(x-1)} dx = \int \left(-\frac{1}{3} \cdot \frac{1}{x+2} + \frac{1}{3} \cdot \frac{1}{x-1} \right) dx$$

$$= -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x-1} dx$$

$$\int \frac{1}{x+2} dx; \quad u=x+2 \\ \downarrow \\ \int \frac{1}{x+2} dx = \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|x+2| + C$$

$$= -\frac{1}{3} \ln|x+2| + \frac{1}{3} \ln|x-1| + C$$

$$= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C \quad \times$$

Ex $\int \frac{2x^2 - 3x + 4}{(x+1)(x-2)^2} dx$

~~$\frac{13}{3 \times 2} = \frac{Q}{3} + \frac{Q}{2}$~~

Ans $\frac{2x^2 - 3x + 4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$8-6+4$ $2x^2 - 3x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

$2x^2 - 3x + 4 = A(\underline{x^2} - \underline{4x} + \underline{4}) + B(\underline{x^2} - \underline{x} - \underline{2}) + C(\underline{x} + \underline{1})$

$2x^2 - 3x + 4 = (A+B)\underline{x^2} + (-4A-B+C)\underline{x} + (4A-2B+C)$

Now $2 = A+B$ — (1)

$-3 = -4A - B + C$ — (2)

$4 = 4A - 2B + C$ — (3)

(3)-(2); $7 = +8A - B$ — (4)

(1)+(4); $9 = 9A \Rightarrow A = 1$

$\therefore B = 1$

$\therefore C = 2$

$\therefore \frac{2x^2 - 3x + 4}{(x+1)(x-2)^2} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}$

$\therefore \int \frac{2x^2 - 3x + 4}{(x+1)(x-2)^2} dx = \int \left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2} \right) dx$

$= \ln|x+1| + \ln|x-2| + 2\left(-\frac{1}{x-2}\right) + C$

$= \ln|x+1| + \ln|x-2| - \frac{2}{x-2} + C$ ~~XXXX~~

(no) $\int \frac{1}{(x-2)^2} dx$

$u = x-2$
 $= \int \frac{1}{u^2} du = -\frac{1}{u} + C$
 $= -\frac{1}{x-2} + C$

การหาอนุพันธ์

$$\frac{Ax^2+Bx+C}{\text{♡}} \quad \text{ตัวนี้ ♡ men mth 7/75}$$

Ex x^2+1

$$\frac{p(x)}{(x^2+1)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{(x^2+1)}$$

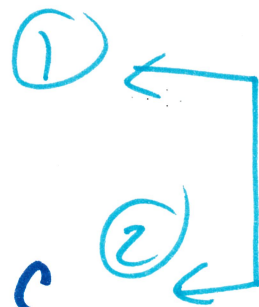
$$\frac{p(x)}{x^2(5x^2+1)^2(x-3)^2} = \boxed{\frac{A}{x} + \frac{B}{x^2}} + \boxed{\frac{C}{x-3} + \frac{D}{(x-3)^2}} + \boxed{\frac{Ex+F}{5x^2+1} + \frac{Gx+H}{(5x^2+1)^2}}$$

$$x^3+x^2+x+1 = 0$$

$$(x-a_1)(Bx^2+Cx+D) = 0$$

$$\frac{1}{(x^2-4)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2}$$

$$\frac{1}{(x-2)(x+2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1}$$



$$\underline{\underline{Ex}} \quad \int \frac{3x^2+x-2}{(x-1)(x^2+1)} dx$$

ឃើញ $\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

ពង្រីក $(x-1)(x^2+1)$ ត្រូវបាន

$$3x^2+x-2 = A(x^2+1) + (Bx+C)(x-1)$$

$$3x^2+x-2 = Ax^2 + A + (Bx^2 - Bx + Cx - C)$$

$$3x^2 + x - 2 = (A+B)x^2 + (-B+C)x + (A-C)$$

រក្សា រង្វង់.

$$A+B = 3 \quad \text{--- ①}$$

$$-B+C = 1 \quad \text{--- ②}$$

$$A - C = -2 \quad \text{--- ③}$$

$$B+C = 5 \quad \text{--- ④}$$

$$\text{①} - \text{③}$$

$$\text{②} + \text{④}$$

$$2C = 6 \rightarrow C = 3$$

$$\therefore B = 2, A = 1$$

$$\therefore \int \frac{3x^2+x-2}{(x-1)(x^2+1)} dx = \int \left(\frac{1}{x-1} + \frac{2x+3}{x^2+1} \right) dx$$

$$= \ln|x-1| + \int \frac{2x}{x^2+1} dx + \int \frac{3}{x^2+1} dx + C.$$

$$\text{ឬ} \int \frac{2x+3}{x^2+1} dx = \int \frac{2x}{x^2+1} dx + \int \frac{3}{x^2+1} dx \quad \#$$

$$u = x^2+1; du = 2x dx$$

$$= \int \frac{1}{u} du = \ln|u| + C \quad 3 \arctan(x)$$

$$= \ln|x^2+1|$$

Ex 7.17 $\int \frac{x+4}{x^2(x^2+4)} dx$

$$\frac{x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} \quad \left\{ \begin{array}{l} A = \frac{1}{4} \\ B = -1 \\ C = -\frac{1}{4} \\ D = -1 \end{array} \right.$$

Ex 7.18 $\int \frac{x^3 - 4x}{(x^2+1)^2} dx$

$$\frac{x^3 - 4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \left\{ \begin{array}{l} A = 1 \\ B = 0 \\ C = -5 \\ D = 0 \end{array} \right.$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{a^2+x^2} dx$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\rightarrow \int \frac{1}{x^2+4} dx = \int \frac{1}{\cancel{x^2+4} \cdot \frac{4}{4}} dx$$

$$= \frac{1}{4} \int \frac{1}{\frac{x^2}{4} + 1} dx$$

$$u = \frac{x}{2} ; du = \frac{1}{2} dx$$

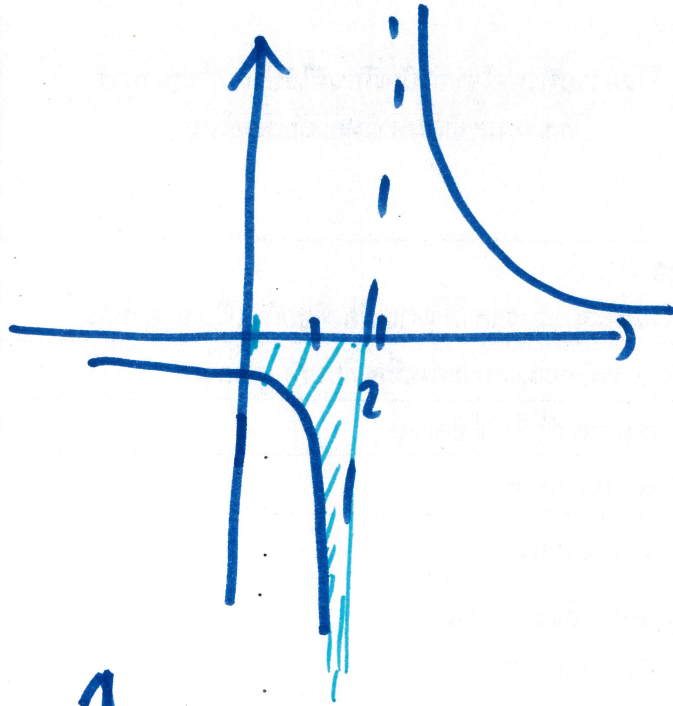
$$= \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx$$

$$= \frac{1}{2} \int \frac{1}{u^2 + 1} du$$

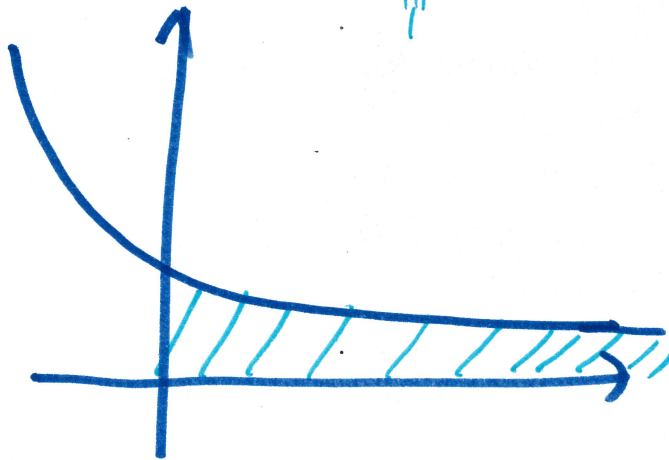
$$= \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$$

$$\int_0^1 (x^2 + 2x + 1) dx$$

$$\textcircled{1} \int_0^2 \frac{1}{x-2} dx$$



$$\textcircled{2} \int_0^{+\infty} e^{-x} dx$$



improper
integral

ចំណាត់ថ្នាក់នៃអាំងតេក្រាលមិនត្រឹមត្រូវ

① Type 1 ចំនុចខាងស្តាំនៃអាំងតេក្រាលគឺជាចំនុចគ្រប់គ្រង

② Type 2 ចំនុចខាងស្តាំនៃអាំងតេក្រាលគឺជាចំនុចគ្រប់គ្រង
នៃអាំងតេក្រាល

③ Type 3 Type ① + Type ②.

Ex $\int_0^{+\infty} \frac{1}{1-x^2} dx$



• အကဲခတ်မှုကိစ္စကို

သိရအောင် $\frac{1}{1-x^2} : \lim_{x \rightarrow 1^+} f(x) = -\infty$

$\parallel$
 $\frac{1}{(1-x)(1+x)}$

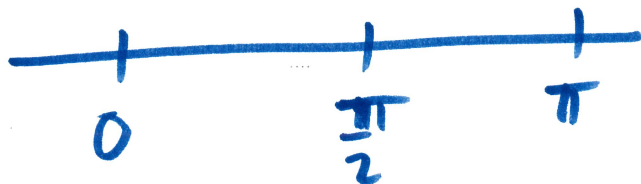
$\lim_{x \rightarrow -1^+} f(x) = +\infty$

• သိရအောင် $x=1$ နှစ်ဘက်ကလေးပါး

→ Type 3

Ex $\int_{-\infty}^{+\infty} x^3 dx \rightarrow \underline{\text{Type 1}}$

Ex $\int_0^{\pi} \sec^2 \theta d\theta \rightarrow \underline{\text{Type 2}}$



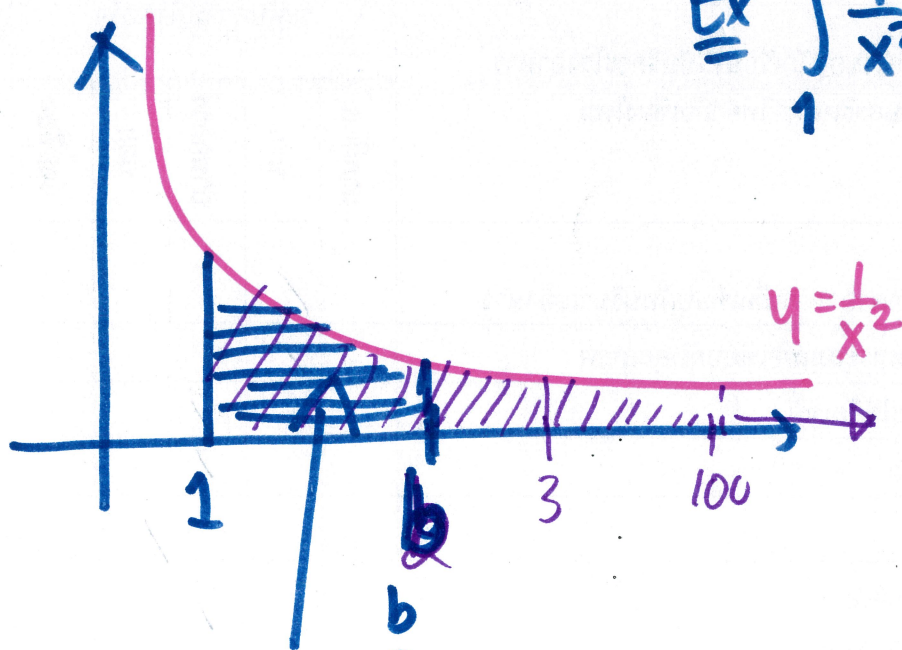
Ex $\int_{-3}^3 \frac{x}{x^2+x+1} dx$

သိရအောင် $x^2+x+1 = (x^2+x+\frac{1}{4}) + \frac{3}{4}$
 $= (x+\frac{1}{2})^2 + \frac{3}{4} > 0$

Table improper Integral

Improper Integral 1

$$\text{Ex } \int_1^{\infty} \frac{1}{x^2} dx$$

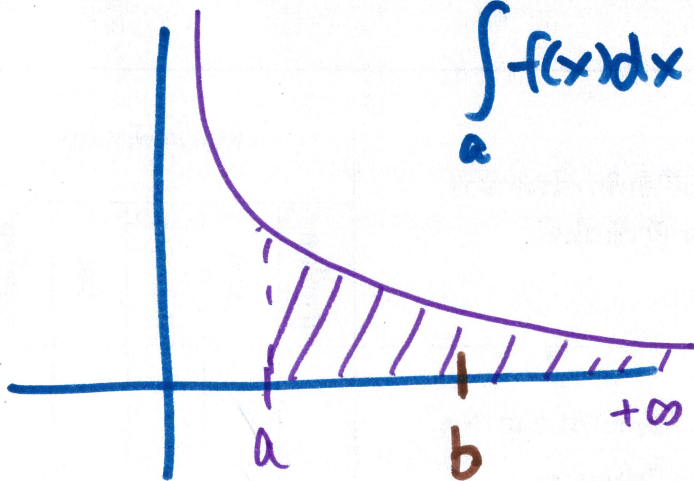


$$A = \int_1^b f(x) dx$$

$$A = \lim_{b \rightarrow +\infty} \int_1^b f(x) dx$$

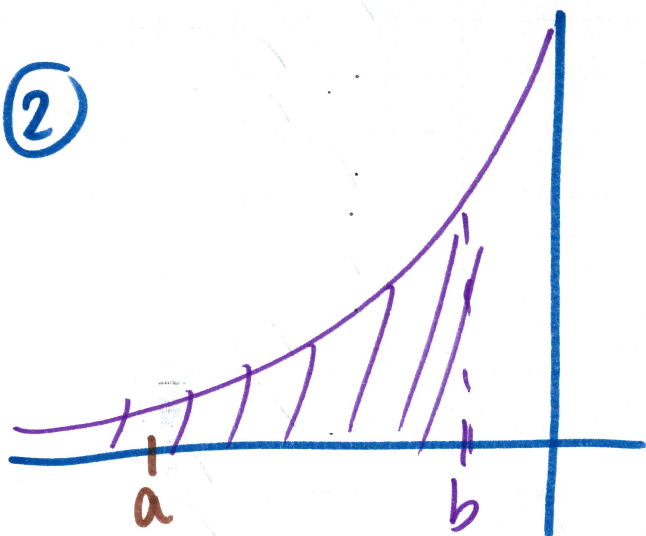
ถ้า A มีค่าเป็นจำนวนจริง $\int_1^{\infty} f(x) dx$ converges
ถ้า A ไม่มีค่าเป็นจำนวนจริง $\int_1^{\infty} f(x) dx$ diverges.

①



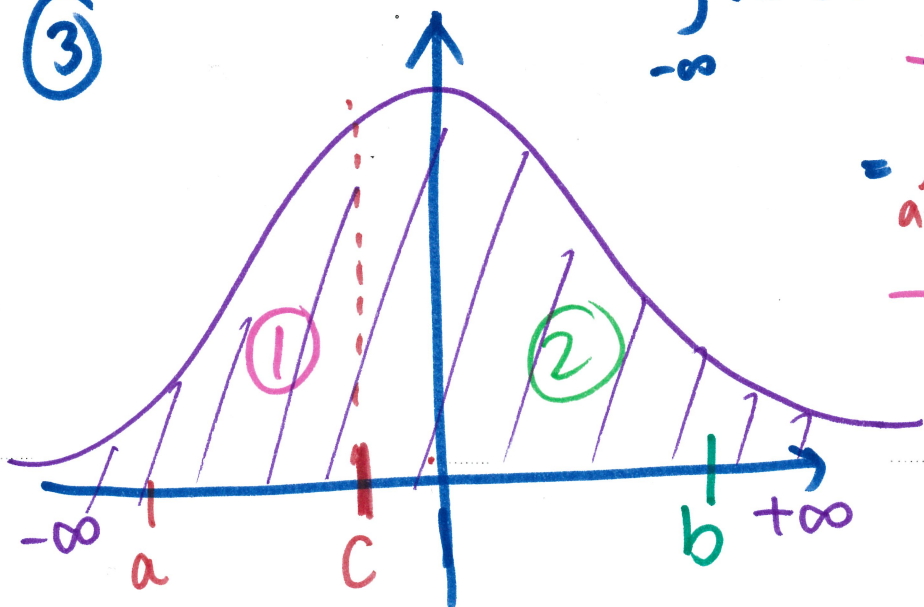
$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

②



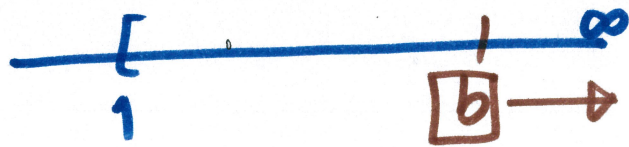
$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

③



$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx \\ &= \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \\ &= \lim_{b \rightarrow +\infty} \int_c^b f(x) dx \end{aligned}$$

Ex $\int_1^{+\infty} \frac{1}{x^3} dx$



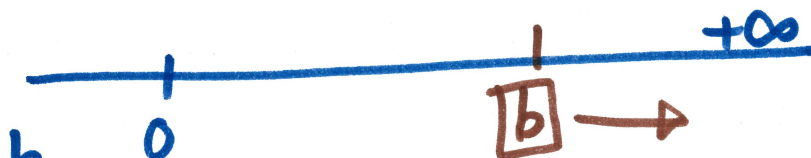
$$\therefore \int_1^{+\infty} \frac{1}{x^3} dx = \lim_{b \rightarrow +\infty} \int_1^b \frac{1}{x^3} dx$$

$$= \lim_{b \rightarrow +\infty} \left[-\frac{1}{2} \cdot \frac{1}{x^2} \right]_1^b$$

$$= \lim_{b \rightarrow +\infty} \left[-\frac{1}{2b^2} + \frac{1}{2} \right] = \frac{1}{2}$$

$$\therefore \int_1^{+\infty} \frac{1}{x^3} dx \text{ converges to } \frac{1}{2} \quad \#$$

Ex $\int_0^{+\infty} \cos x \, dx$



$$\therefore \int_0^{+\infty} \cos x \, dx = \lim_{b \rightarrow +\infty} \int_0^b \cos x \, dx$$

$$= \lim_{b \rightarrow +\infty} [\sin x]_0^b$$

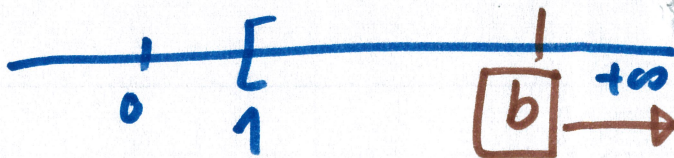
$$= \lim_{b \rightarrow +\infty} [\sin b - \sin 0]$$

$$= \lim_{b \rightarrow +\infty} (\sin b) \quad \text{DNE.}$$

diverges.



~~Ans~~ $\int_1^{+\infty} \frac{\ln x}{x} dx$



$$\therefore \int_1^{+\infty} \frac{\ln x}{x} dx = \lim_{b \rightarrow +\infty} \int_1^b \frac{\ln x}{x} dx$$

$$= \lim_{b \rightarrow +\infty} \left. \frac{(\ln x)^2}{2} \right|_1^b$$

$$= \lim_{b \rightarrow +\infty} \left[\frac{(\ln b)^2}{2} - \frac{(\ln 1)^2}{2} \right]$$

$$= \lim_{b \rightarrow +\infty} \left[\frac{(\ln b)^2}{2} \right]$$

$$= +\infty$$

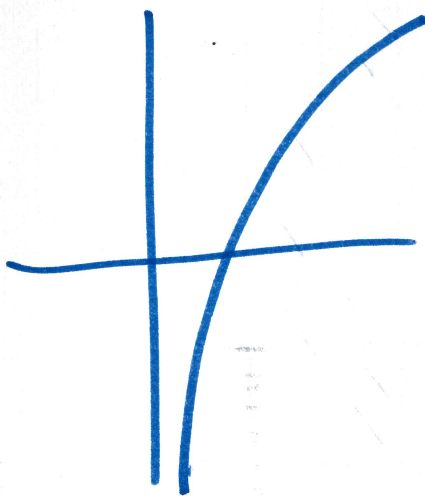
$$\therefore \int_1^{+\infty} \frac{\ln x}{x} dx \text{ diverges.}$$

$$\int \frac{\ln x}{x} dx$$

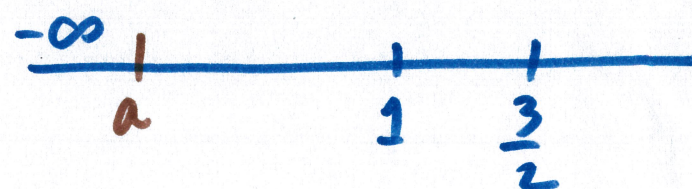
Let $u = \ln x$
 $du = \frac{1}{x} dx$

$$\int \frac{\ln x}{x} dx = \int u du$$

$$= \frac{u^2}{2} + C$$

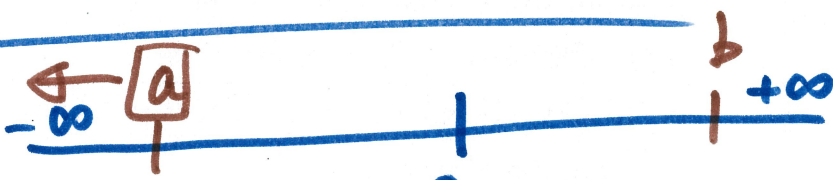


Ex $\int_{-\infty}^1 \frac{1}{3-2x} dx$



$$\int_{-\infty}^1 \frac{1}{3-2x} dx = \lim_{a \rightarrow -\infty} \int_a^1 \frac{1}{3-2x} dx$$

Ex $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$



$$\therefore \int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \int_{-\infty}^0 \frac{x}{x^2+1} dx + \int_0^{+\infty} \frac{x}{x^2+1} dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 \frac{x}{x^2+1} dx + \lim_{b \rightarrow +\infty} \int_0^b \frac{x}{x^2+1} dx$$