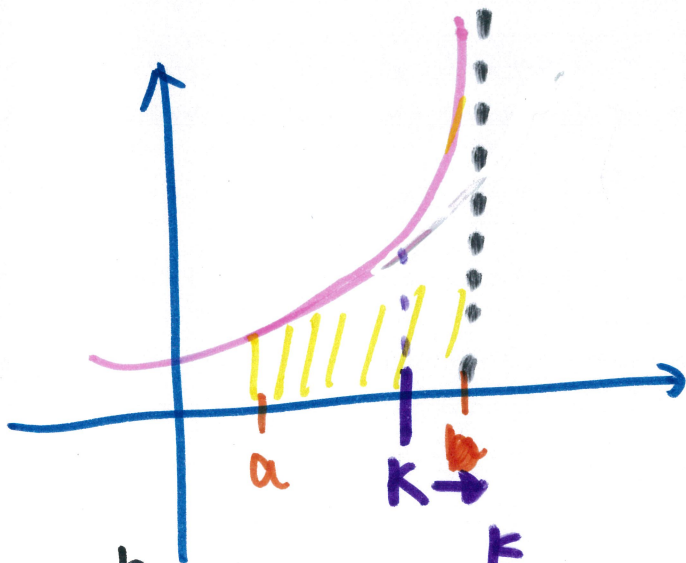
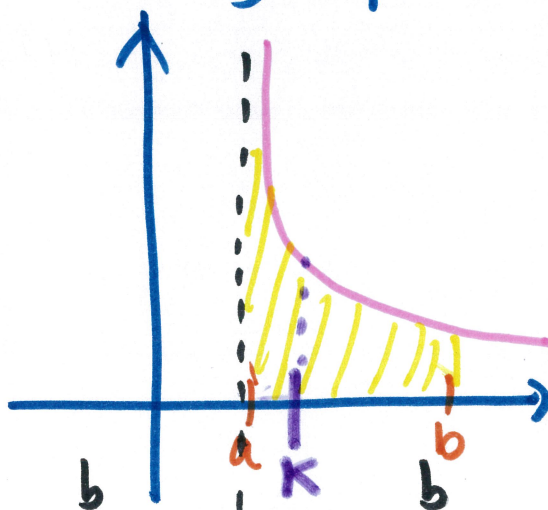


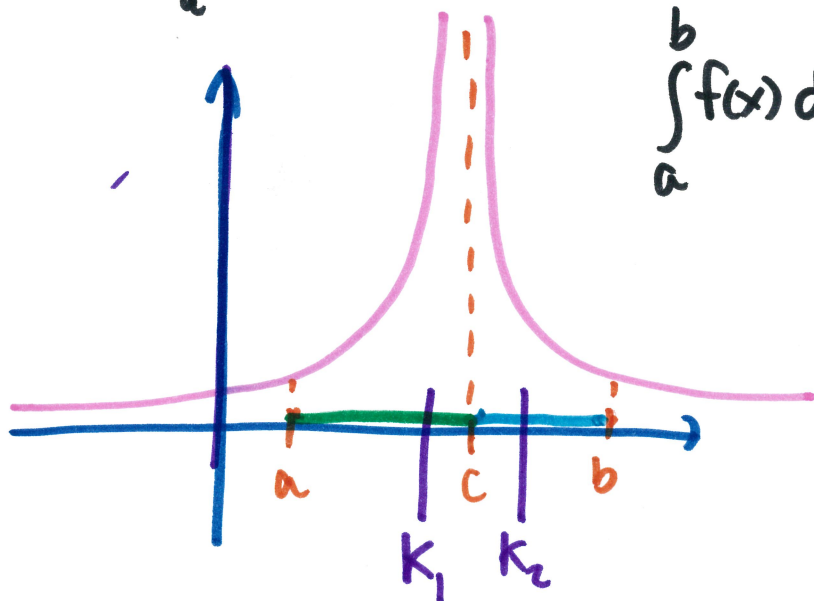
# Improper Integral Type 2 $\rightarrow$ infinity of $a$



$$\int_a^b f(x) dx = \lim_{K \rightarrow b^-} \int_a^K f(x) dx$$

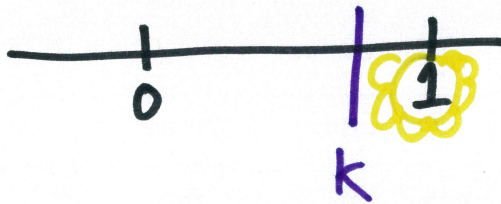


$$\int_a^b f(x) dx = \lim_{K \rightarrow a^+} \int_K^b f(x) dx$$



$$\begin{aligned} \int_a^b f(x) dx &= \boxed{\int_a^c f(x) dx} + \boxed{\int_c^b f(x) dx} \\ &= \lim_{K_1 \rightarrow c^-} \int_a^{K_1} f(x) dx + \lim_{K_2 \rightarrow c^+} \int_{K_2}^b f(x) dx \end{aligned}$$

Ex  $\int_0^1 \frac{1}{1-x} dx$



$$\int_0^1 \frac{1}{1-x} dx = \lim_{k \rightarrow 1^-} \int_0^k \frac{1}{1-x} dx$$

$$= \lim_{k \rightarrow 1^-} [-\ln|1-k| + \ln 1]$$

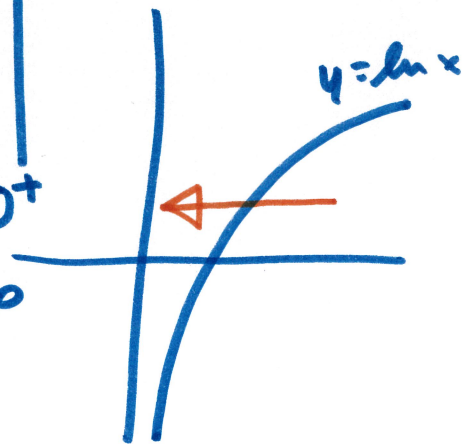
note  $k \rightarrow 1^-$  note  $1-k \rightarrow 0^+$

$$\therefore \ln|1-k| \rightarrow -\infty$$

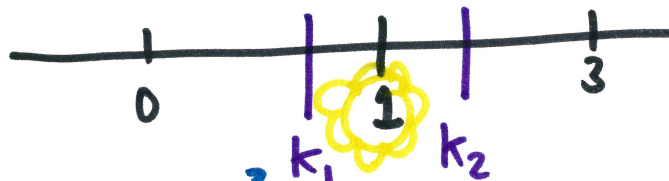
$$= +\infty$$

$\therefore$  diverges.

$$\int \frac{1}{1-x} dx = \ln|1-x| + C$$



Ex  $\int_0^3 \frac{dx}{(x-1)^{2/3}}$



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^{k_1} \frac{dx}{(x-1)^{2/3}} + \int_1^{k_2} \frac{dx}{(x-1)^{2/3}} + \int_{k_2}^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{k_1 \rightarrow 1^-} \int_0^{k_1} \frac{dx}{(x-1)^{2/3}} + \lim_{k_2 \rightarrow 1^+} \int_{k_2}^3 \frac{dx}{(x-1)^{2/3}}$$

Answer  $\int \frac{dx}{(x-1)^{2/3}} \quad u = x-1; du = dx$

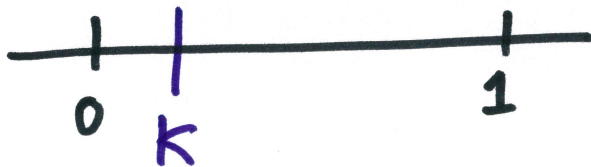
$$= \int \frac{du}{u^{2/3}} = 3u^{1/3} + C = 3(x-1)^{1/3} + C$$

$$\lim_{k_1 \rightarrow 1^-} [3(k_1-1)^{1/3} - 3(-1)^{1/3}] = 3$$

$$\lim_{k_2 \rightarrow 1^+} [3(3-1)^{1/3} - 3(k_2-1)^{1/3}] = 3\sqrt[3]{2}$$

Converges.

Ex (a)  $\int_0^1 (\ln x)^2 dx$



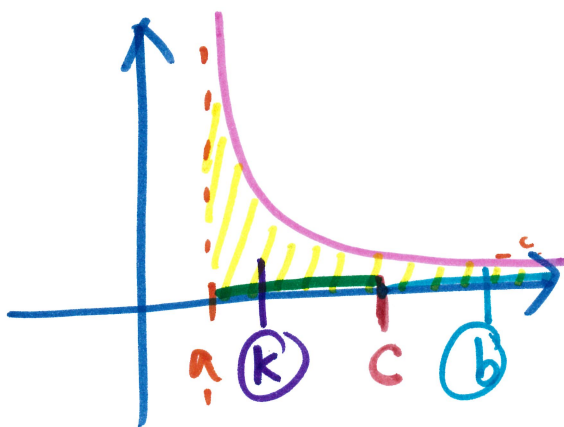
$$\therefore \int_0^1 (\ln x)^2 dx = \lim_{k \rightarrow 0^+} \int_k^1 (\ln x)^2 dx$$

(d)  $\int_{-\infty}^{\infty} \frac{1}{x^2+3} dx$



$$\begin{aligned} \therefore \int_{-\infty}^{\infty} \frac{1}{x^2+3} dx &= \int_{-\infty}^0 \frac{1}{x^2+3} dx + \int_0^{+\infty} \frac{1}{x^2+3} dx \\ &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{x^2+3} dx + \lim_{b \rightarrow +\infty} \int_0^b \frac{1}{x^2+3} dx \end{aligned}$$

Improper integral Type 3 = Type 1 + Type 2

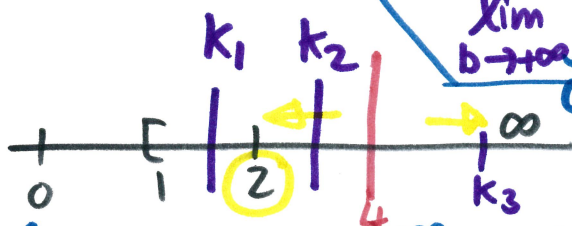


$$\int_a^{+\infty} f(x) dx = \boxed{\int_a^c f(x) dx} + \boxed{\int_c^{\infty} f(x) dx}$$

Type 2                      Type 1

$$= \lim_{k \rightarrow a^+} \int_k^c f(x) dx + \lim_{b \rightarrow +\infty} \int_c^b f(x) dx$$

Ex (c)  $\int_1^{\infty} \frac{1}{x(x-2)} dx$



$$\begin{aligned} &= \int_1^2 \frac{1}{x(x-2)} dx + \int_2^4 \frac{1}{x(x-1)} dx + \int_4^{+\infty} \frac{1}{x(x-2)} dx \\ &= \lim_{k_1 \rightarrow 2^-} \int_1^{k_1} \frac{1}{x(x-2)} dx + \lim_{k_2 \rightarrow 2^+} \int_{k_2}^4 \frac{1}{x(x-1)} dx + \lim_{k_3 \rightarrow +\infty} \int_4^{k_3} \frac{1}{x(x-2)} dx \end{aligned}$$



Ex  $\int_0^{+\infty} \frac{1}{x} dx$

$$\therefore \int_0^{+\infty} \frac{1}{x} dx = \boxed{\int_0^1 \frac{1}{x} dx} + \boxed{\int_1^{+\infty} \frac{1}{x} dx}$$

$$= \lim_{k_1 \rightarrow 0^+} \int_{k_1}^1 \frac{1}{x} dx + \lim_{k_2 \rightarrow +\infty} \int_1^{k_2} \frac{1}{x} dx$$

with (2)  $\lim_{k_1 \rightarrow 0^+} \int_{k_1}^1 \frac{1}{x} dx = \lim_{k_1 \rightarrow 0^+} [\ln|x|]_{k_1}^1$

$$= \lim_{k_1 \rightarrow 0^+} [\ln 1 - \ln |k_1|]$$

$$= +\infty$$

$$\therefore \int_0^{+\infty} \frac{1}{x} dx \text{ diverges.}$$


---



# Differential Equations (DEs)

$$x^2 + 2x + 1 = 0$$

$$\boxed{x = -1}$$

- $\frac{dy}{dx} = x^2$

หา y ที่ทำให้  $\frac{dy}{dx} = x^2 \rightarrow y = \frac{x^3}{3} + C$

- $\frac{dy}{dx} = \sin x \cos y$

สมมติว่า  $y = f(x)$  ที่สอดคล้องกับ  $\frac{dy}{dx} = \sin x \cos y$ .

• สมการเชิงอนุพันธ์ : คือ สมการที่หาความสัมพันธ์ของฟังก์ชัน  $y$  และ  $y'$

$$\frac{dy}{dx} = x^2$$

↓  
อันดับ (order) คือ อันดับสูงสุดของอนุพันธ์ในสมการ

Ex  $\frac{dy}{dx} = \cos x$       order 1

$y - y' = (\ln x + \tan 2x)y'' + \sqrt{x} + y$       order 2

• คำตอบของสมการ → คำตอบทั่วไป (general solution)  
ถ้าให้เงื่อนไขที่จุดใดจุดหนึ่ง (เช่น  $y(0) = 1$ )  
คำตอบทั่วไป จะเปลี่ยนเป็น คำตอบเฉพาะ (particular solution)  
ข้อมูลที่ให้ไว้ตอนแรก เงื่อนไขเริ่มต้น (initial condition)

[illegible]

Ex  $\boxed{\frac{dy}{dx} = 0; y=1}$  ~~หาค่า~~ ให้ค่าของ  $\frac{dy}{dx} = 0$

Ex  $\boxed{\frac{dy}{dx} = y}$ ;  $y = e^x$  is a solution or not?

$$\therefore \frac{dy}{dx} = \frac{d(e^x)}{dx} = e^x = y \quad (\checkmark)$$

$y = Ce^x \leftarrow C$  ជាអថេរ  $\int$  តើមានអថេរ  $x$  ដែរ?

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = ce^x = y \quad (\checkmark)$$

Ex  $x \frac{dy}{dx} + y = 2x$ ;  $y = x + \frac{1}{x}$  is a particular solution?

$$\therefore x \frac{dy}{dx} + y = x \left(1 - \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)$$

$$= x - \frac{1}{x} + x + \frac{1}{x}$$
$$= 2x \quad (\checkmark)$$

$y = x + \frac{C}{x}$  υπερβολικός αλγόριθμος?

$$\therefore x \frac{dy}{dx} + y = x \left( 1 - \frac{C}{x^2} \right) + \left( x + \frac{C}{x} \right)$$

$$= x - \frac{c}{x} + x + \frac{c}{x}$$

$$= 2x \quad (\checkmark)$$



Ex 8.1

$$\left. \begin{array}{l} y_1 = e^x \\ y_2 = e^{-x} \\ y_3 = c_1 e^x + c_2 e^{-x} \end{array} \right\} \begin{array}{l} \text{คำตอบของสมการ} \\ y'' = y \text{ หรือไม่?} \end{array}$$

$$\underline{y_1 = e^x} : \begin{array}{l} y_1' = e^x \\ \boxed{y_1'' = e^x = y_1} \end{array} \quad \text{✓}$$

$$\underline{y_2 = e^{-x}} : \begin{array}{l} y_2' = -e^{-x} \\ \boxed{y_2'' = e^{-x} = y_2} \end{array}$$

$$\underline{y_3 = c_1 e^x + c_2 e^{-x}} : \begin{array}{l} y' = c_1 e^x - c_2 e^{-x} \\ \boxed{y'' = c_1 e^x + c_2 e^{-x}} \end{array} \quad \begin{array}{l} \checkmark \\ \checkmark \end{array}$$

Ex  $y'' = -y$  สมการของตัว  $y = c_1 \cos x + c_2 \sin x$

$$\downarrow$$
$$y(0) = 1, y\left(\frac{\pi}{2}\right) = 2$$

$$\rightarrow y(0) = c_1 \cos(0) + c_2 \sin(0) = 1$$
$$\therefore \boxed{c_1 = 1}$$

$$\rightarrow y\left(\frac{\pi}{2}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 2$$
$$\therefore \boxed{c_2 = 2}$$

$$\therefore \text{คำตอบคือ } y = \cos x + 2 \sin x$$

$$(\text{ตรวจสอบ } y(0) = 1, y\left(\frac{\pi}{2}\right) = 2)$$



Ex 8.2

$$2y \frac{dy}{dx} = 3x^2 ; y(0) = -3$$

คำตอบทั่วไปคือ  $y^2 = x^3 + C$   
 หาว่าคำตอบเฉพาะคือ  $y(0) = -3$  คืออะไร?

$$C = 9$$

# 8.1 Separable Equation (สมการที่แยกตัวแปรได้)

$$\boxed{h(y) \frac{dy}{dx} = g(x)}$$

เป็นฟังก์ชันของ y
เป็นฟังก์ชันของ x

|                                               | Separable                                     | $h(y)$                   | $g(x)$            |
|-----------------------------------------------|-----------------------------------------------|--------------------------|-------------------|
| Ex $\frac{dy}{dx} = \frac{x}{y} \Rightarrow$  | $\boxed{y} \frac{dy}{dx} = \boxed{x}$         | $y$                      | $x$               |
| $x \frac{dy}{dx} = -y^2 \Rightarrow$          | $\frac{1}{y^2} \frac{dy}{dx} = -\frac{1}{x}$  | $\frac{1}{y^2}$          | $-\frac{1}{x}$    |
| $\frac{dy}{dx} = y - \frac{y}{x} \Rightarrow$ | $\frac{1}{y} \frac{dy}{dx} = 1 - \frac{1}{x}$ | $\frac{1}{y}$            | $1 - \frac{1}{x}$ |
| $\frac{dy}{dx} = y(1 - \frac{1}{x})$          |                                               |                          |                   |
| $\frac{y}{x} \frac{dy}{dx} = x\sqrt{1+y^2}$   | $\frac{y}{\sqrt{1+y^2}} \frac{dy}{dx} = x^2$  | $\frac{y}{\sqrt{1+y^2}}$ | $x^2$             |
| $\frac{x^2}{1+y^2} \frac{dy}{dx} = 0$         | $(1+y^2) \frac{dy}{dx} = x^2$                 | $1+y^2$                  | $x^2$             |

Ex

$$\frac{dy}{dx} = -4xy^2 \rightarrow \frac{1}{y^2} \frac{dy}{dx} = -4x$$

$$y(0) = 1$$

$$\frac{1}{y^2} dy = (-4x) dx$$

อินทิเกรตทั้งสองข้าง

$$\int \frac{1}{y^2} dy = \int (-4x) dx$$

$$-\frac{1}{y} = -2x^2 + C$$

บวก C  
ทันที

$$y = \frac{1}{2x^2 - C}$$

นำมอด  
ให้ลงตัว.

$$\text{ถ้า } y(0) = 1; \quad 1 = \frac{1}{0 - C} \Rightarrow C = -1$$

$$\therefore y = \frac{1}{2x^2 + 1} \leftarrow \text{นำมอด  
ให้ลงตัว}$$

Ex

$$(4y - \cos y) \frac{dy}{dx} - 3x^2 = 0; \quad y(0) = 0$$

$$(4y - \cos y) \frac{dy}{dx} = 3x^2$$

$$(4y - \cos y) dy = 3x^2 dx$$

$$\int (4y - \cos y) dy = \int 3x^2 dx$$

$$2y^2 - \sin y = x^3 + C \leftarrow \text{นำมอดให้ลงตัว}$$

$$2(y(x))^2 - \sin(y(x)) = x^3 + C.$$

$$\text{ถ้า } x=0 \text{ แล้ว } y(0)=0$$

$$2(y(0))^2 - \sin(y(0)) = 0^3 + C$$

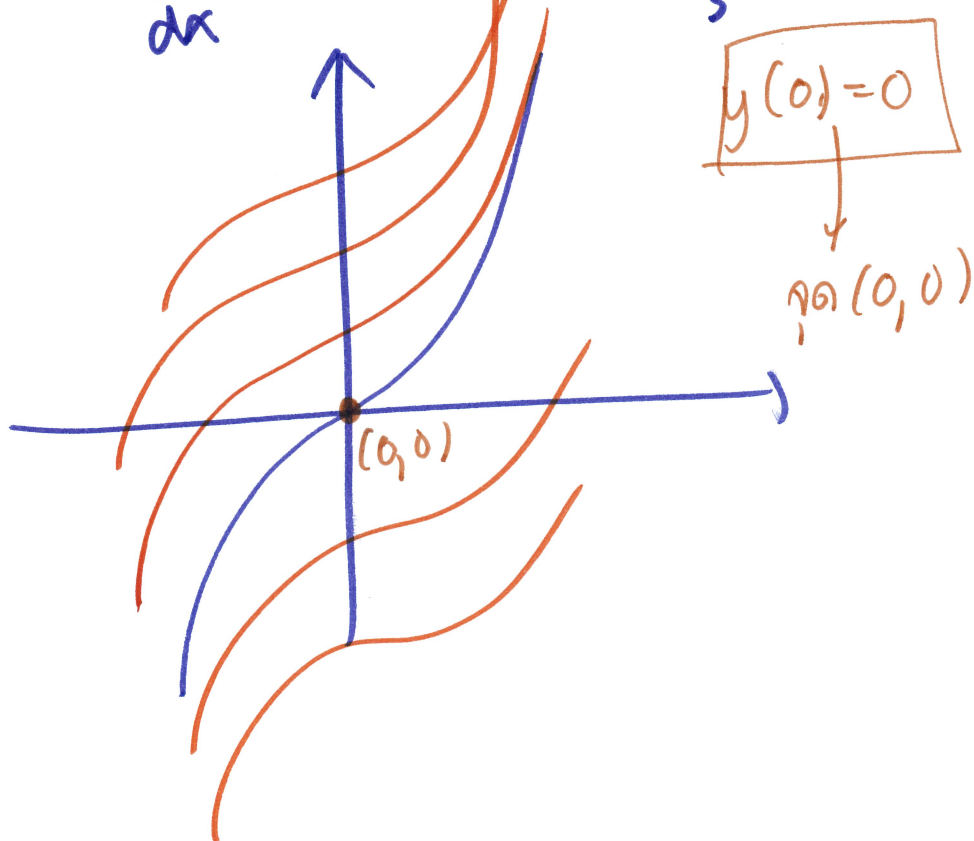
$$C = 0$$

$$\text{นำมอดให้ลงตัว: } \boxed{2y^2 - \sin y = x^3}$$

$\left(\frac{dy}{dx}\right)$  = gradient at point  $(x, y)$  on  $\rightarrow$   
 $x^2y + xy^2 = 4$



$\frac{dy}{dx} = x^2 \rightarrow y = \frac{x^3}{3} + C$





Ex จงหาเส้นโค้งในระนาบ  $xy$  ที่ผ่านจุด  $(0,3)$  และมีความชันที่จุด  $(x,y)$  เท่ากับ  $\frac{2x}{y^2}$

$$\boxed{\frac{dy}{dx} = \frac{2x}{y^2}} \rightarrow \text{initial condition } \boxed{y(0) = 3}$$

Sol<sup>n</sup>

$$y^2 \frac{dy}{dx} = 2x$$

$$y^2 dy = 2x dx$$

$$\int y^2 dy = \int 2x dx$$

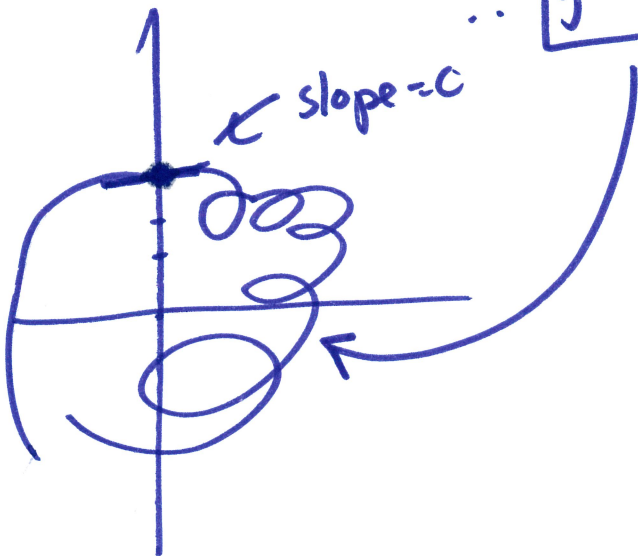
$$\frac{y^3}{3} = x^2 + C$$

$$\therefore \boxed{y^3 = 3x^2 + 3C}$$

$$\because y(0) = 3 \quad ; \quad 27 = 3(0)^2 + 3C$$

$$\therefore C = 9$$

$$\therefore \boxed{y^3 = x^2 + 27}$$



บท. เสริม (improper integral Type 3)

จงตรวจสอบว่าอินทิกรัลต่อไปนี้ converge หรือ diverge

$$(1) \int_{-\infty}^{-2} \frac{2}{x^2-1} dx$$

$$(4) \int_{-\infty}^{\infty} \frac{1}{x^2} dx$$

$$(2) \int_0^{+\infty} \frac{dx}{(1+x)\sqrt{x}}$$

$$(5) \int_0^{+\infty} \frac{\ln x}{x^2} dx$$

$$(3) \int_{-\infty}^0 \frac{1}{x(x-2)} dx$$

$$(6) \int_{-\infty}^{+\infty} \frac{x}{(x^2-4)^{3/2}} dx$$

$$\textcircled{1} \int \frac{1}{\sqrt{4-x^2}} dx, \int \frac{x}{\sqrt{4-x^2}} dx, \int \frac{x^2}{\sqrt{4-x^2}} dx$$

$$\textcircled{2} \int \frac{x-1}{x^2-2x-3} dx, \int \frac{x}{x^2-2x-3} dx, \int \frac{x}{x^2-2x+3} dx$$

$$\textcircled{3} \int \frac{e^{2x}}{e^{2x}-2e^x-3} dx.$$

## 8.2 First-order Linear Equation သဘောတရားနှင့် ဝိသေသ 1

သုည. ဝက်

$\frac{dy}{dx}$  မှာ  $y$  မှာ  $\frac{dy}{dx}$  မှာ

$$\frac{dy}{dx} + p(x)y = q(x)$$

Ex

Linear

$p(x)$

$q(x)$

$$\frac{dy}{dx} + x^2 y = e^x$$

✓

$$x^2$$

$$e^x$$

$$y' - 3e^x y = e^x$$

✓

$$-3e^x$$

$$e^x$$

$$x \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$0$$

$$\frac{1}{x}$$

$$xy' + 2y = 4x^2$$

$$y' + \frac{2}{x}y = 4x$$

$$\frac{2}{x}$$

$$4x$$

$$\frac{dy}{dx} + y - \frac{1}{1+e^x} = 0$$

$$\frac{dy}{dx} + y = \frac{1}{1+e^x}$$

$$1$$

$$\frac{1}{1+e^x}$$

✓



$$\frac{dy}{dx} + p(x)y = q(x) \quad \text{--- (*)}$$

ရှာ  $\mu(x)$  ၅ဲး  $\mu(x)$  ကံးကံးကံး .

$$\mu(x) \frac{dy}{dx} + \mu(x) p(x) y = \mu(x) q(x)$$

အဲးကံး  $\frac{d(\mu(x)y)}{dx} = \mu(x) \frac{dy}{dx} + y \frac{d\mu(x)}{dx}$

မံးကံး

$$\mu(x) p(x) y = \frac{d\mu(x)y}{dx}$$

↓

$$\frac{d\mu(x)}{dx} = \mu(x) p(x)$$

$$\frac{1}{\mu(x)} d\mu(x) = p(x) dx$$

$$\int \frac{1}{\mu(x)} d\mu(x) = \int p(x) dx$$

$$\ln |\mu(x)| = \int p(x) dx$$

$$\therefore \mu(x) = e^{\int p(x) dx}$$

$$\frac{d(\mu(x)y)}{dx} = \mu(x) q(x)$$

အဲးကံး: integrating factor

(ကံးကံးကံး separable ဝဲး)

Ex

$$\frac{dy}{dx} - y = e^{2x}$$

$$\cdot p(x) = -1$$

$$\cdot \mu(x) = e^{\int -1 dx} = e^{-x}$$

คูณทั้งสองข้าง  $\mu(x)$  ทั้งสองข้าง

$$e^{-x} \left[ \frac{dy}{dx} - y \right] = e^{-x} \cdot e^{2x}$$

$$\frac{d}{dx} [e^{-x} y] = e^x$$

$$d(e^{-x} y) = e^x dx$$

อินทิเกรตทั้งสองข้าง

$$\int d(e^{-x} y) = \int e^x dx$$

$$e^{-x} y = e^x + C$$

$$\boxed{y = e^{2x} + C e^x}$$

$$y = e^{2x} + C e^x$$

Ex

$$x \frac{dy}{dx} + 2y = x^2 - x + 1$$

$$\frac{dy}{dx} + \boxed{\frac{2}{x}} y = x - 1 + \frac{1}{x}$$

$P(x)$

$$\therefore \mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln|x|} = x^2$$

คูณทั้งสองข้าง  $\mu(x)$  ทั้งสองข้าง.

~~$$x^2 \left( x \frac{dy}{dx} + 2y \right) =$$~~

$$\frac{d}{dx} (\mu(x)y) \rightarrow \overset{\mu(x)}{x^2} \left[ \frac{dy}{dx} + \frac{2}{x} y \right] = x^3 - x^2 + x$$

$$\frac{d}{dx} (x^2 y) = x^3 - x^2 + x$$

$$d(x^2 y) = (x^3 - x^2 + x) dx$$

$$\int d(x^2 y) = \int (x^3 - x^2 + x) dx$$

$$x^2 y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\boxed{y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}}$$



Ex

$$x \frac{dy}{dx} - y = x \quad ; \quad y(1) = 2$$

$$\frac{dy}{dx} - \frac{1}{x} y = 1$$

$$\therefore \mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\frac{1}{x} \left[ \frac{dy}{dx} - \frac{1}{x} y \right] = \frac{1}{x}$$

$$\frac{d}{dx} \left( \frac{1}{x} y \right) = \frac{1}{x}$$

$$d\left(\frac{1}{x} y\right) = \frac{1}{x} dx$$

$$\int d\left(\frac{1}{x} y\right) = \int \frac{1}{x} dx$$

$$\frac{1}{x} y = \ln|x| + C$$

$$\boxed{y = x \ln|x| + Cx}$$

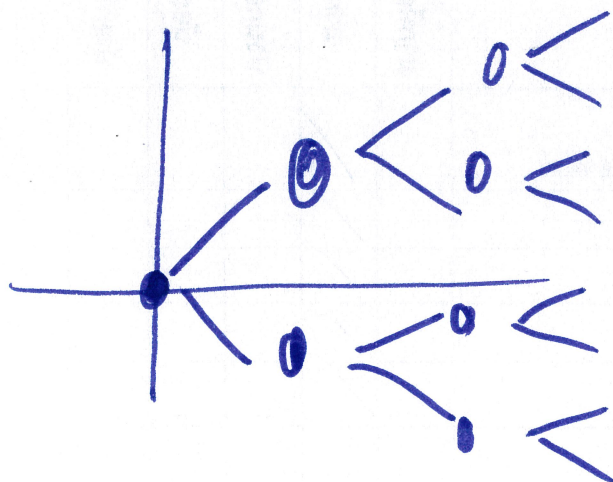
$$y(1) = 2 \quad ; \quad 2 = 1 \ln|1| + C(1)$$

$$C = 2$$

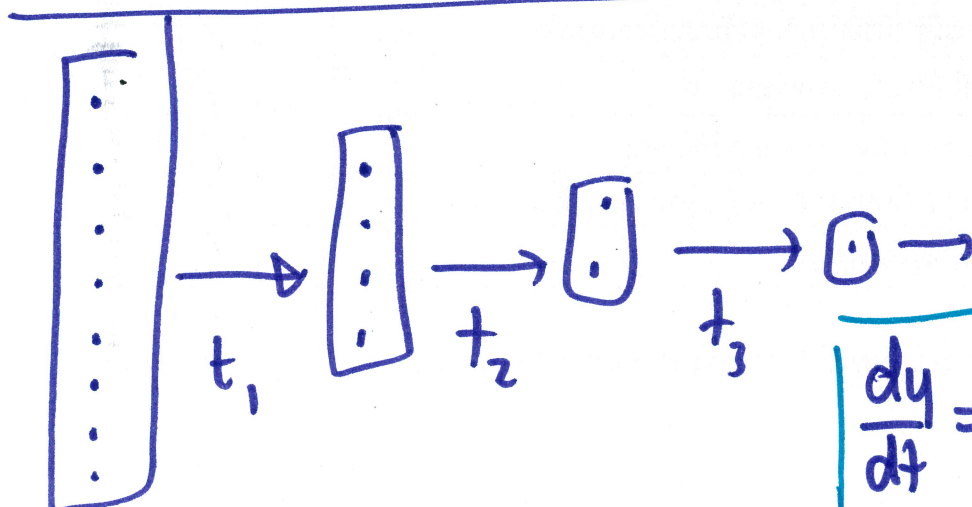
Answer:  $\boxed{y = x \ln x + 2x}$

## Example of applications.

### Exponential Growth / Decay Model



$$\frac{dy}{dt} = ky \quad ; k > 0$$



$$\frac{dy}{dt} = -ky; k > 0$$

$$\frac{dy}{dt} = -ky; y(0) = y_0$$

$$\hookrightarrow \frac{dy}{dt} = -ky$$
$$\int dy = \int -k dt$$

$$\frac{1}{y} \frac{dy}{dt} = -k$$

$$\therefore \frac{1}{y} dy = -k dt$$

$$\int \frac{1}{y} dy = \int -k dt$$

$$\ln |y| = -kt + C$$

$$y = e^{-kt+C} = e^{-kt} \cdot e^C$$



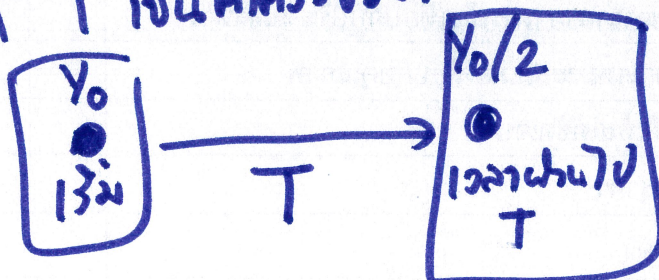
an particular solution

$$y_0 = Ce^{-k(0)} \Rightarrow C = y_0$$

$$\therefore y = y_0 e^{-kt}$$

Ex 8.8

ใน T หนึ่งหน่วยครึ่งชีวิต  $\rightarrow$  ค่า k ในสมการ T



แก้สมการ \* (ใช้สมการครึ่งชีวิต T ในสมการ)

$$\frac{y_0}{2} = y_0 e^{-kT}$$

$$\therefore \frac{1}{2} = e^{-kT} \Rightarrow \ln \left| \frac{1}{2} \right| = \ln(e^{-kT})$$

$$-\ln 2 = -kT$$

$$\therefore T = \frac{\ln 2}{k}$$

ถ้า Radium-226 ครึ่งชีวิต 1600 ปี

$$1600 = \frac{\ln 2}{k}$$

$$\therefore k = \frac{\ln 2}{1600}$$