ทบทวน Integration by parts.

Ex $\int x^2 \sin x \, dx$

เลือก $u = x^2$ $\quad dv = \sin x \, dx$

$\therefore du = 2x \, dx \; ; \quad v = \int \sin x \, dx = -\cos x$

$$\therefore \int x^2 \sin x \, dx = -x^2 \cos x - \int (2x)(-\cos x)\, dx$$
$$= -x^2 \cos x + 2\underline{\int x \cos x \, dx} \quad \text{—} \circledast$$

$u_1 = x \; ; \quad dv_1 = \cos x \, dx$

$du_1 = dx ; \quad v_1 = \sin x$

พิจารณา $\int x \cos x \, dx = x \sin x - \int \sin x \, dx$
$$= x \sin x + \cos x$$

$$\circledast = -x^2 \cos x + 2x \sin x + 2\cos x + C \quad \#$$

Technique 2 การอินทิเกรตฟังก์ชันตรีโกณบางแบบ

เอกลักษณ์ตรีโกณ : $\boxed{\sin^2 x + \cos^2 x = 1}$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

$\tan^2 x + 1 = \sec^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Ex ① ตัวถูกอินทิเกรต มี $\sin x, \cos x$ อยู่ด้วยกัน $\rightarrow \int \sin^{m} x \cos^{n} x \, dx$

$\rightarrow m$ odd (เลขคี่)

$\rightarrow n$ odd (เลขคี่)

$\rightarrow m, n$ even (เลขคู่)

Ex 7.6 $\int \sin^3 x \, dx$

$$\sin^2 x + \cos^2 x = 1$$
$$\boxed{\sin^2 x = 1 - \cos^2 x}$$

จาก $\int \sin^3 x \, dx = \int \sin x \cdot \sin^2 x \, dx$

$$= \int \sin x \, (1 - \cos^2 x) \, dx$$

ให้ $u = \cos x$

$du = -\sin x \, dx$

$\therefore \boxed{dx = -\frac{1}{\sin x} du}$

$$= \int \cancel{\sin x} \cdot (1 - u^2) \frac{du}{-\cancel{\sin x}}$$

$$= \int (u^2 - 1) \, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C \quad \#$$

หลักการ
1. ดูว่าใครเป็นตัวกำลังเลขคี่
2. แยกเทอมกำลังเลขคี่ออกเป็น กำลัง 1 กับกำลังเลขคู่
3. ใช้เอกลักษณ์ตรีโกณ กำลังสองมาช่วย
4. ใช้การอินทิเกรตแบบเปลี่ยนตัวแปร

Ex 7.7 จงหา $\int \cos^5 x \, dx$

$$\sin^2 x + \cos^2 x = 1$$
$$\boxed{\cos^2 x = 1 - \sin^2 x}$$

$$\int \cos^5 x \, dx = \int \cos x \cdot \cos^4 x \, dx$$

$$= \int \cos x \, (\cos^2 x)^2 \, dx$$

ให้ $u = \sin x \,;\ du = \cos x \, dx$

$\therefore dx = \frac{du}{\cos x}$

$$= \int \cos x \, (1 - \sin^2 x)^2 \, dx$$

$$= \int \cancel{\cos x} \, (1 - u^2)^2 \frac{du}{\cancel{\cos x}}$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C \quad \#$$

Ex 7.8 $\int \sin^2 x \cos^3 x \, dx$

$\boxed{\sin^2 x + \cos^2 x = 1}$

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$$
$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$
$du = \cos x \, dx$
$\therefore dx = \frac{du}{\cos x}$

$$= \cancel{\int \sin^2 x (}$$
$$= \int u^2 (1 - u^2) \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$
$$= \int u^2 (1 - u^2) \, du$$
$$= \int (u^2 - u^4) \, du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \quad \#$$

Ex $\int \sin^2 x \cos^2 x \, dx$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$2x$ $4x$

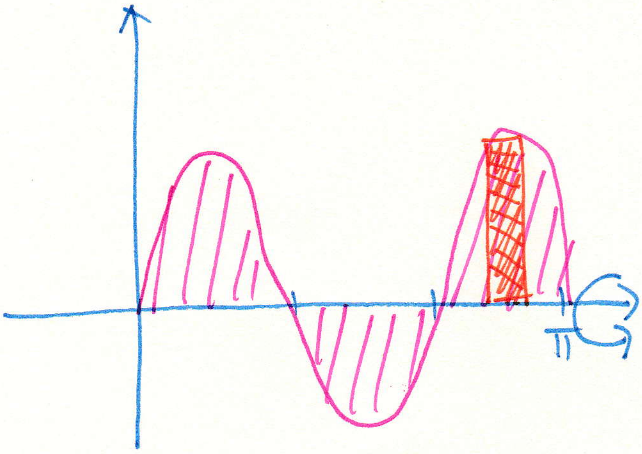
$$\therefore \int \sin^2 x \cos^2 x \, dx = \int \left[\frac{1}{2}(1 - \cos 2x)\right] \cdot \left[\frac{1}{2}(1 + \cos 2x)\right] dx$$
$$= \frac{1}{4} \int [(1 - \cos 2x)(1 + \cos 2x)] \, dx$$
$$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx.$$
$$= \frac{1}{4} \int dx - \frac{1}{4} \int \cos^2 2x \, dx$$
$$= \frac{1}{4} x - \frac{1}{4} \left[\int \frac{1}{2} (1 + \cos 4x) \, dx \right]$$
$$= \frac{1}{4} x - \frac{1}{8} \left[\int dx + \int \cos 4x \, dx \right]$$
$$= \frac{1}{4} x - \frac{x}{8} - \frac{1}{8} \cdot \frac{\sin 4x}{4} + C$$

$\int \cos 4x \, dx$
ให้ $u = 4x$; $du = 4 dx$
$\therefore dx = \frac{du}{4}$
$\therefore \int \cos 4x \, dx$
$= \int \cos u \frac{du}{4}$
$= \frac{\sin u}{4} + C$
$= \frac{\sin 4x}{4} + C$

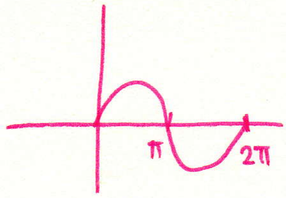
จำ

$$\boxed{\begin{aligned}\int \sin(mx)\,dx &= -\frac{\cos mx}{m}+C\\ \int \cos(mx)\,dx &= \frac{\sin mx}{m}+C.\end{aligned}}\ \Bigg\}\ \text{(m เป็นตัวเลข)}$$

Ex 7.9 จงหาปริมาตรของ V ที่เกิดจากการหมุน $y=\sin 3x$ บน $[0,\pi]$ หมุนรอบแกน x



($y=\sin x$



$$V=\int_0^{\pi}\pi\,[\sin 3x]^2\,dx$$

$$=\pi\int_0^{\pi}(\sin^2 3x)\,dx$$

$$\boxed{\sin^2\heartsuit=\frac{1}{2}(1-2\cos\heartsuit)}$$

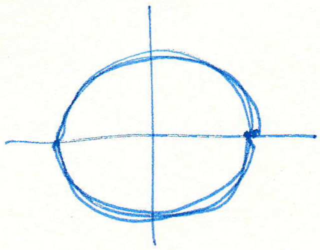
$$\therefore\ V=\pi\int_0^{\pi}\sin^2 3x\,dx=\pi\int_0^{\pi}\left[\frac{1}{2}-\frac{1}{2}\cos 6x\right]dx=\frac{\pi}{2}\int_0^{\pi}(1-\cos 6x)\,dx \quad \circledast$$

พิจารณา $\displaystyle\int(1-\cos 6x)\,dx = x-\frac{\sin 6x}{6}+C$

$$\circledast\ V=\frac{\pi}{2}\left[x-\frac{\sin 6x}{6}\right]_0^{\pi}$$

$$=\frac{\pi}{2}\left[\pi-\cancel{\frac{\sin 6\pi}{6}}^{\,0}-0+\cancel{\frac{\sin 0}{6}}\right]$$

$$=\frac{\pi^2}{2}$$



$\int \sin mx \cos nx \, dx$ / $\int \sin mx \sin nx \, dx$ / $\int \cos mx \cos nx \, dx$

1. $\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$
2. $\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$
3. $\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

Ex 7.10 $\int \sin 3x \cos 5x \, dx$

$$\int \sin 3x \cos 5x \, dx = \int \frac{1}{2}[\sin(3x+5x) + \sin(3x-5x)]dx$$

$$= \frac{1}{2}\int [\sin(8x) + \sin(-2x)]dx$$

$\cos(-x) = \cos x$
$\sin(-x) = -\sin x$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} - \frac{\cos(-2x)}{(-2)}\right] + C$$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} + \frac{\cos(-2x)}{2}\right] + C$$

$$= -\frac{\cos 8x}{16} + \frac{\cos(2x)}{4} + C$$

Ex เพิ่มเติม $\int \sin 2x \sin x \, dx$

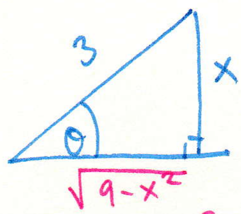
$$\int \sin 2x \sin x \, dx = \int \frac{1}{2}[\cos(2x-x) - \cos(2x+x)]dx$$

$$= \frac{1}{2}\int [\cos x - \cos 3x]dx$$

$$= \frac{1}{2}\left[\sin x - \sin \frac{3x}{3}\right] + C$$

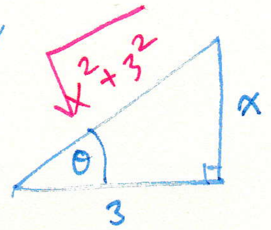
Technique 3 การอินทิเกรต แบบแทนค่าตรีโกณมิติ

Ex $\sqrt{a^2-x^2}$ / $\sqrt{a^2+x^2}$ / a^2+x^2 / $\sqrt{x^2-a^2}$



$\sin\theta = \frac{3}{x}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$



$\sin\theta = \frac{x}{\sqrt{x^2+3^2}}$

$\cos\theta = \frac{3}{\sqrt{x^2+3^2}}$

(x ตัวแปร, a ตัวเลข)

FORM

1. $\sqrt{a^2-x^2} \Rightarrow$ เปลี่ยน $x = a\sin\theta \Rightarrow$
2. $\sqrt{a^2+x^2} \Rightarrow$ เปลี่ยน $x = a\tan\theta \Rightarrow$
3. $\sqrt{x^2-a^2} \Rightarrow$ เปลี่ยน $x = a\sec\theta \Rightarrow$

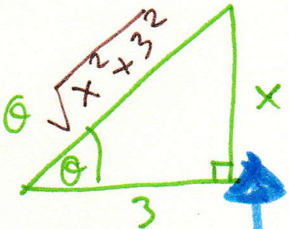
} จัดรูปโดยใช้ ความสัมพันธ์ ของตรีโกณฯ

Ex $\int \frac{dx}{\sqrt{9+x^2}}$ คล้ายกับ FORM 2 ①

$\sqrt{3^2+x^2}$

เปลี่ยนแทน : $x = 3\tan\theta \Rightarrow \frac{x}{3} = \tan\theta$

$\therefore dx = 3\sec^2\theta\, d\theta$ ②



$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+(3\tan\theta)^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}}$$

$1+\tan^2\theta = \sec^2\theta$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\sqrt{\boxed{1+\tan^2\theta}}}$$

$$= \int \frac{\cancel{3}\sec^2\theta\, d\theta}{\cancel{3}\sqrt{\sec^2\theta}}$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$ ③

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$ ④ //

SARUB

① ดูว่า เข้า FORM ไหน → ② $x = \sim\sim\sim$, $dx = \sim\sim\sim\, d\theta$, เปลี่ยน x เป็นเทอม θ → ③ ใช้ trigone จัดรูป + อินทิเกรต → ④ ใช้ Δ มุมฉากหา ความสัมพันธ์ในเทอม x

Ex 7.12 $\int \frac{x^3}{\sqrt{9-x^2}}dx$ คล้ายกับ FORM 1

$\sqrt{9-x^2} \to \sqrt{3^2-x^2}$

เปลี่ยนแทน $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{9-x^2}}dx = \int \frac{(3\sin\theta)^3}{\sqrt{9-9\sin^2\theta}} \cdot 3\cdot\cos\theta\, d\theta$$

* แปลงทุกอย่างให้เป็นฟังก์ชันตรีโกณฯ

$$= \int \frac{3^3\sin^3\theta \cdot \cancel{3}\cdot\cos\theta\, d\theta}{\cancel{3}\sqrt{1-\sin^2\theta}}$$

$1-\sin^2\theta = \cos^2\theta$

$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$$= 3^3\int \frac{\sin^3\theta \cdot \cancel{\cos\theta}}{\cancel{\cos\theta}}d\theta$$

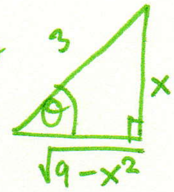
$$= 27\int \sin^3\theta\, d\theta$$ ← (ดูข้อ ต.ย. เก่า)

$\vdots$

$$= 27\left[\frac{\cos^3\theta}{3} - \cos\theta\right] + C$$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$



$$= 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - \cancel{27}^{9}\,\frac{\sqrt{9-x^2}}{\cancel{3}_1} + C$$

$$= \frac{1}{3}\left(\sqrt{9-x^2}\right)^3 - 9\sqrt{9-x^2} + C$$ #

Ex 7.11 $\int \frac{dx}{x^2\sqrt{4x^2-3}}$ คล้าย FORM 3

$\sqrt{4x^2-3} \to \sqrt{(2x)^2-(\sqrt{3})^2}$

ให้ $\boxed{2x} = \sqrt{3}\sec\theta \Rightarrow 4x^2 = 3\sec^2\theta$

$x = \frac{\sqrt{3}}{2}\sec\theta$

$\therefore dx = \frac{\sqrt{3}}{2}\sec\theta\tan\theta\, d\theta$

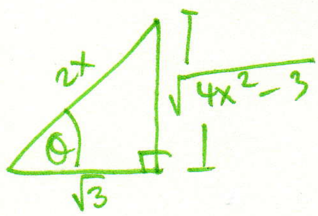
$$\therefore \int \frac{dx}{x^2\sqrt{4x^2-3}} = \int \frac{\frac{\sqrt{3}}{2}\sec\theta\tan\theta\, d\theta}{\left(\frac{\sqrt{3}}{2}\sec\theta\right)^2\sqrt{3\sec^2\theta - 3}} = \int \frac{\frac{\sqrt{3}}{2}\sec\theta\tan\theta\, d\theta}{\left(\frac{\sqrt{3}}{2}\right)^2\sec^2\theta\cdot\sqrt{3}\,\underbrace{\sqrt{\sec^2\theta - 1}}_{\tan^2\theta}}$$

$$= \int \frac{\cancel{\frac{\sqrt{3}}{2}}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\left(\frac{\sqrt{3}}{2}\right)^2\sec^{\cancel{2}}\theta\cdot\cancel{\sqrt{3}}\cdot\cancel{\tan\theta}} = \int \frac{\frac{1}{\cancel{2}}\,1\,d\theta}{\frac{3}{\cancel{4}\,2}\sec\theta}$$

$2x = \sqrt{3}\sec\theta$

$\sec\theta = \frac{2x}{\sqrt{3}}$

$\cos\theta = \frac{\sqrt{3}}{2x}$



$$= \frac{2}{3}\int \frac{1}{\sec\theta}d\theta = \frac{2}{3}\int \cos\theta$$

$$= \frac{2}{3}\sin\theta + C = \frac{2}{3}\left[\frac{\sqrt{4x^2-3}}{2x}\right] + C$$