หลักการ

FORM 1 $\sqrt{a^2-u^2} \rightarrow u = a\sin\theta$

FORM 2 $\sqrt{a^2+u^2} \rightarrow u = a\tan\theta$

FORM 3 $\sqrt{u^2-a^2} \rightarrow u = a\sec\theta$

Ex 7.13 $\int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}\,dx$ ให้ $A = e^x \therefore dA = e^x dx \Rightarrow dx = \frac{dA}{e^x}$

$$\therefore \int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}\,dx = \int \frac{\cancel{A}}{\sqrt{A^2+A+1}}\,\frac{dA}{\cancel{A}} = \int \frac{1}{\sqrt{A^2+A+1}}\,dA$$

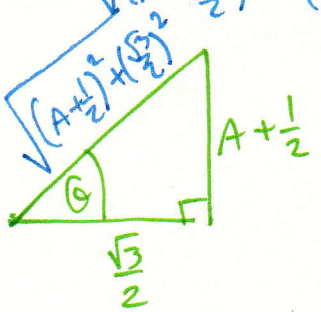
$$= \int \frac{dA}{\sqrt{\left(A^2+2(A)\left(\frac{1}{2}\right)+\frac{1}{4}\right)-\frac{1}{4}+1}}$$

$$= \int \frac{dA}{\sqrt{\left(A^2+A+\frac{1}{4}\right)+\frac{3}{4}}} = \int \frac{dA}{\sqrt{\left(A+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$A+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$ $\quad \tan^2\theta+1 = \sec^2\theta$

$\therefore dA = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$\therefore \int \frac{dA}{\sqrt{\left(A+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\sqrt{\left(\frac{\sqrt{3}}{2}\tan\theta\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} = \frac{\frac{\sqrt{3}}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\int \frac{\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}}$$



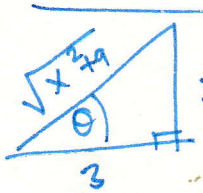
$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{(A+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{\frac{\sqrt{3}}{2}} + \frac{A+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| + C$$

$$= \ln\left|\frac{2}{\sqrt{3}}\sqrt{A^2+A+1} + \frac{2A+1}{\sqrt{3}}\right| + C$$

$$= \ln\left|\frac{2}{\sqrt{3}}\sqrt{e^{2x}+e^x+1} + \frac{2e^x+1}{\sqrt{3}}\right| + C \quad \#$$



... อินทิเกรตโดยเทียบรูป $\theta + C$

$\rightarrow \sin\theta = \frac{x}{\sqrt{x^2+9}} \Rightarrow \theta = \arcsin\left(\frac{x}{\sqrt{x^2+9}}\right)$

Technique 4 Partial fraction.

ฟังก์ชันตรรกยะ (rational function) $f(x) = \frac{P(x)}{Q(x)}$ → พหุนาม, → พหุนาม.

Ex $f(x) = \frac{x^2+2x+4}{(x^2-1)(x+1)}$ $P(x)$: พหุนาม $\deg(P(x)) = 2$; $Q(x)$: พหุนาม $\deg(Q(x)) = 3$

$f(x) = \frac{x^3+4x^2-x+1}{x^2+1}$ $P(x)$, $Q(x)$ $\deg(P(x)) = 3$, $\deg(Q(x)) = 2$

ในการแยกเศษส่วนย่อย จะทำได้ก็ฟังก์ชันตรรกยะอยู่ในรูป เศษส่วนแท้

$f(x) = \frac{P(x)}{Q(x)}$ ← ฟังก์ชันตรรกยะแท้ (proper rational f^{n}) $\deg(P(x)) < \deg(Q(x))$

Ex $f(x) = \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}$ $P(x)$, $Q(x)$ $\deg(P(x)) = 4$, $\deg(Q(x)) = 2$

$$\begin{array}{r} 3x^2+1 \\ x^2+x-2 \overline{)\,3x^4+3x^3-5x^2+x-1} \\ \pm 3x^4 \pm 3x^3 \mp 6x^2 \\ \hline x^2+x-1 \\ \pm x^2 \pm x \mp 2 \\ \hline 1 \end{array}$$

ตัวตั้ง = ตัวหาร × ผลหาร + เศษ

ตัวตั้ง ← $(3x^2+1)$ ผลหาร

$\therefore f(x) = (3x^2+1) + \frac{1}{x^2+x-2}$ ← เศษ

CASE 1 ตัวส่วนมีตัวประกอบเชิงเส้น $(Ax+b)$

$$\frac{1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$\frac{1}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\frac{x^2+1-1}{(2x+1)^3(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+1} + \frac{D}{(2x+1)^2} + \frac{E}{(2x+1)^3}$$

Ex 7.14 $\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x+2)(x-1)}$

พิจารณา $\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$ —⊛

คูณ ⊛ ด้วย $(x-2)(x+1)$

$$1 = A(x-1) + B(x+2)$$
$$1 = Ax - A + Bx + 2B$$
$$0x+1 = (A+B)x + (-A+2B)$$

เทียบ สปส $\quad A+B = 0 \quad -A+2B = 1$

$\therefore -(-B)+2B = 1$

$3B = 1 \rightarrow B = \frac{1}{3}$

$\therefore A = -\frac{1}{3}$

$$\frac{1}{(x+2)(x-1)} = -\frac{1}{3}\cdot\frac{1}{(x+2)} + \frac{1}{3}\cdot\frac{1}{x-1}$$

$$\therefore \int \frac{1}{(x+2)(x-1)}\,dx = \int -\frac{1}{3}\cdot\frac{1}{x+2}\,dx + \int \frac{1}{3}\cdot\frac{1}{x-1}\,dx$$
$$= -\frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$
$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C \quad \#$$

Ex 7.15 $\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx$

พิจารณา $\frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$2x^2-3x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

$2x^2-3x+4 = A(x^2-4x+4) + B(x^2-x-2) + C(x+1)$

$2x^2-3x+4 = Ax^2-4Ax+4A + Bx^2-Bx-2B + Cx+C$

$2x^2-3x+4 = (A+B)x^2 + (-4A-B+C)x + (4A-2B+C)$

เทียบ สปส.

$A+B=2$ — (1)

$-4A-B+C=-3$ — (2)

$4A-2B+C=4$ — (3)

(3)−(2) $8A-B=7$ — (4)

(1)+(4) $9A=9$

$\therefore A=1$ — $B=1$

(2) $\rightarrow C=-3+4A+B \Rightarrow C=-3+4+1=2$

$\therefore \frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}$

$\therefore \int \frac{2x^2-3x+4}{(x-1)(x-2)^2}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{1}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx$

$= \ln|x+1| + \ln|x-2| + 2\int \frac{1}{(x-2)^2}\,dx$

$= \ln|x+1| + \ln|x-2| - \frac{2}{x-2} + C$

$u = x-2$

$du = dx$

$\therefore \int \frac{1}{(x-2)^2}\,dx$

$= \int \frac{1}{u^2}\,du$

$= -u^{-1} + C$

$= -\frac{1}{x-2} + C$

CASE 2 ตัวส่วนมีตัวประกอบกำลังสอง $(ax^2+bx+c)^m$ โดยที่ $b^2-4ac<0$

↳ แยกตัวประกอบไม่ได้ x^2+1

Ex $$\frac{x+2}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\frac{x^2+2x+4}{(x^2+x+1)^2(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}$$

Ex 7.16 $\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx$

พิจารณา $$\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$\therefore$ $3x^2+x-2 = A(x^2+1)+(Bx+C)(x-1)$

$3x^2+x-2 = Ax^2+A+Bx^2-Bx+Cx-C$

$3x^2+x-2 = (A+B)x^2+(-B+C)x+(A-C)$

ให้ยบ สปส. $A+B=3$ — ①

$-B+C=1$ — ②

$A-C=-2$ — ③

$A+C=4$ — ④

①+②; ③+④; $2A=2 \Rightarrow A=1$

$\Downarrow$

$B=2 \Rightarrow C=3$

$\therefore$ $$\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{2x+3}{x^2+1}$$

$\therefore$ $$\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2x+3}{x^2+1}\,dx \quad \text{—} \otimes$$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx$$

$= \ln|x-1| + \ln|x^2+1| + 3\arctan x + C$

พิจารณา $\int \frac{2x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\therefore dx = \frac{du}{2x}$

$\int \frac{2x}{x^2+1}\,dx = \int \frac{2x}{u}\cdot\frac{du}{2x} = \ln|u| + C = \ln|x^2+1| + C$

Ex 7.17 $\int \frac{x+4}{x^2(x^2+4)}dx \Rightarrow \frac{x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$

$(A=\frac{1}{4}, B=1, C=-\frac{1}{4})$

Ex 7.18 $\int \frac{x^3-4x}{(x^2+1)^2}dx \Rightarrow \frac{x^3-4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$(A=1, B=0, C=-5, D=0)$

$$\frac{1}{(x^2-4)(x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-4} \quad \text{<-}$$

$$\Downarrow$$

$$\frac{1}{(x-2)(x+2)x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2}$$

⊛ check ว่าตัวส่วน แยกตัวประกอบสุดแล้วหรือยัง !