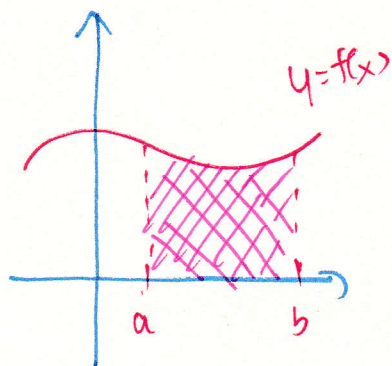
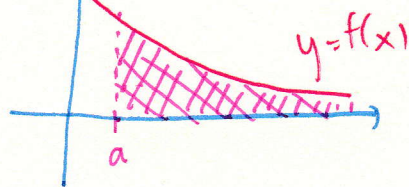


$$\int_a^b f(x) dx$$



$$\int_a^{+\infty} f(x) dx$$



Improper integral 3 kinds.

① Type 1 ขอบเขตการอินทิเกรตเป็นอนันต์

Ex $\int_a^{+\infty} f(x) dx$, $\int_{-\infty}^a f(x) dx$, $\int_{-\infty}^{+\infty} f(x) dx$

② Type 2 ขั้วตกที่ก้นกราฟจนทำให้ f ไม่ต่อเนื่องบนขอบเขต [a, b]

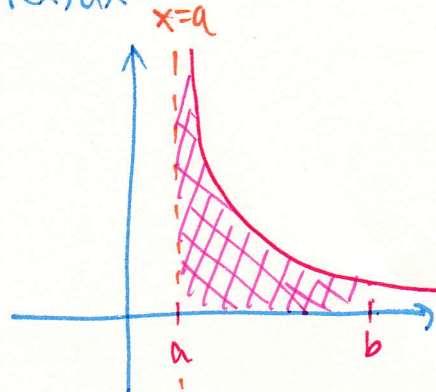
↓
มีเส้นกำกับแนวตั้ง (VA) ในช่วง [a, b]

Ex $\int_a^b f(x) dx$ / $\int_a^b f(x) dx$ / $\int_a^b f(x) dx$
 $x=a$ VA / $x=b$ VA / $c \in [a, b]$ VA

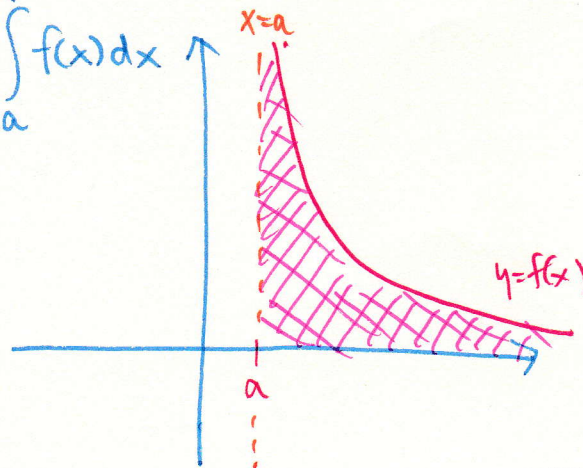
③ Type 3 = ① + ②

$\int_a^{+\infty} f(x) dx$
 $a \leftarrow x=a$ VA.

$$\int_a^b f(x) dx$$



$$\int_a^{+\infty} f(x) dx$$



"Improper integral"
 ขั้วตกที่ก้นกราฟ

7.6 Improper Integrals ปรับปรุง 6/10/2561

It is assumed in the definition of the definite integral $\int_a^b f(x) dx$ that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable.

Our main objective is to extend the concept of definite integrals for infinite intervals of integration and for integrands with vertical asymptotes within the interval of integration. If a function f has a vertical asymptote, then f is said to have an **infinite discontinuity**.

An integral over an infinite interval of integration or an integral with an infinite discontinuity will be called an **improper integral**.

There are three types of improper integrals:

1. Improper integrals with infinite intervals of integration.
2. Improper integrals with infinite discontinuities in the interval of integration.
3. Improper integrals with infinite discontinuities over infinite intervals of integration.

Example 7.20 Determine if each of the following integrals is improper. If so, specify its type.

1. $\int_0^{+\infty} \frac{1}{1-x^2} dx$

2. $\int_{-\infty}^{+\infty} x^3 dx$

3. $\int_0^{\pi} \sec^2 \theta d\theta$

4. $\int_{-3}^{-\infty} \frac{x}{x^2+x+1} dx$

① $\int_0^{+\infty} \frac{1}{1-x^2} dx$ 1. ขอบเขตบนเป็น $+\infty$ ✓
2. $f(x) = \frac{1}{1-x^2}$ ที่ $x=1, -1$ เป็น VA ✓ } เป็น improper integral ชนิดที่ 3

② $\int_{-\infty}^{+\infty} x^3 dx$ 1. ขอบเขตบน $+\infty$, ขอบเขตล่าง $-\infty$ ✓
2. $f(x) = x^3$ เป็นฟังก์ชันพหุนาม ต่อเนื่องทุกจุดใน $\mathbb{R}$ ✗ } เป็น improper int. ชนิดที่ 1

③ $\int_0^{\pi} \sec^2 \theta d\theta$ 1. ขอบเขตบน, ล่าง เป็นค่าจริง ✗
2. $f(\theta) = \sec^2 \theta = \frac{1}{\cos^2 \theta}$ ที่ $\theta = \frac{\pi}{2}$ ทำให้ $f(\theta)$ ขาดค่าได้ ; เกิด VA. ที่ $\theta = \frac{\pi}{2}$ ✓ } เป็น improper int. ชนิดที่ 2.

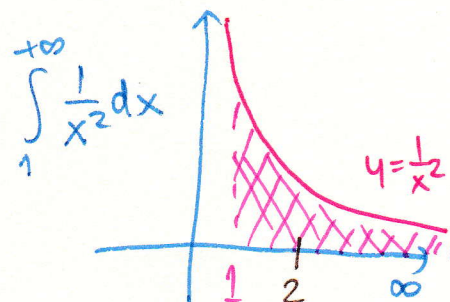
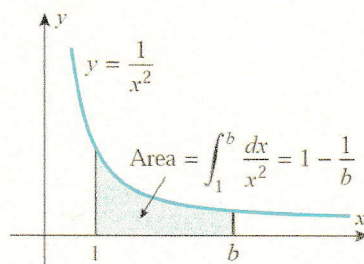
④ $\int_{-3}^3 \frac{x}{x^2+x+1} dx$ 1. ขอบเขตบน, ล่าง เป็นค่าจริง ✗
2. $f(x) = \frac{x}{x^2+x+1}$

$$= \frac{x}{\left(x^2 + 2(x) \cdot \frac{1}{2} + \frac{1}{4}\right) - \frac{1}{4} + 1} = \frac{x}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} > 0$$

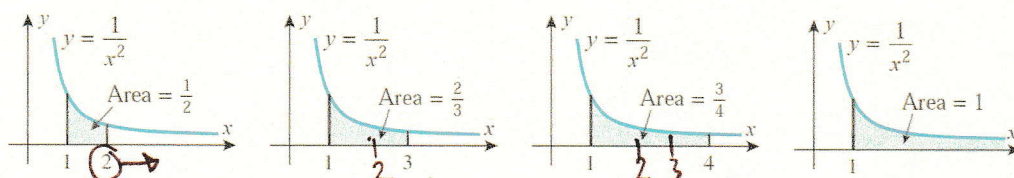
เป็น proper integral (ปรับปรุง 6/10/2561)

7.6.1 Integrals over infinite intervals : Type ① ∫₁[∞] 1/x² dx

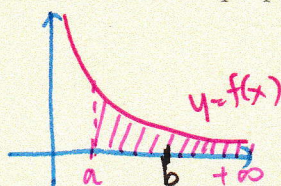
Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x -axis. Let us begin by calculating the portion of the area that lies above a finite interval $[1, b]$, where $b > 1$ is arbitrary. That area is $\int_1^b \frac{dx}{x^2} = 1 - \frac{1}{b}$



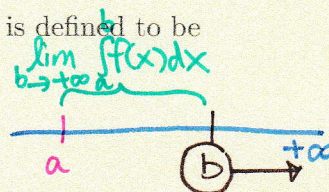
If we now allow b to increase so that $b \rightarrow +\infty$, then the portion of the area over the interval $[1, b]$ will begin to fill out the area over the entire interval $[1, +\infty)$, and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be $A = \int_1^{\infty} \frac{dx}{x^2} = \lim_{b \rightarrow +\infty} \int_1^b \frac{dx}{x^2} = \lim_{b \rightarrow +\infty} (1 - \frac{1}{b}) = 1$



DEFINITION The improper integral of f over the interval $[a, +\infty)$ is defined to be

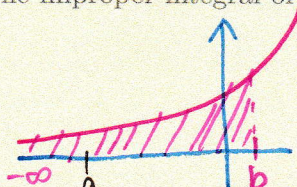


$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx,$$

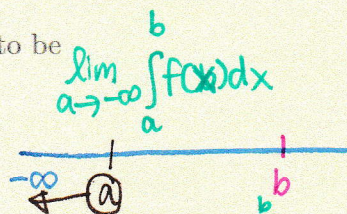


The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, b]$ is defined to be

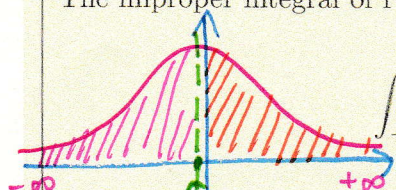


$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx,$$

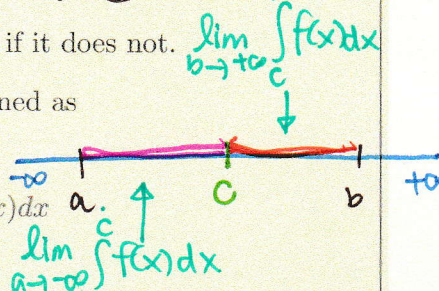


The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, +\infty)$ is defined as



$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$



where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

Example 7.21 Compute $\int_1^{+\infty} \frac{1}{x^3} dx$.

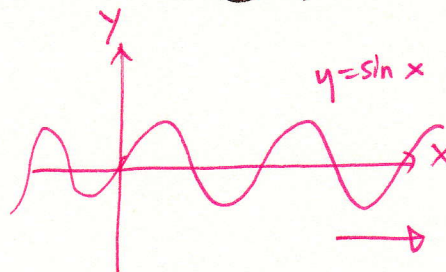
$$\begin{aligned} \int_1^{+\infty} \frac{1}{x^3} dx &= \lim_{b \rightarrow +\infty} \int_1^b \frac{1}{x^3} dx \\ &= \lim_{b \rightarrow +\infty} \left[-\frac{x^{-2}}{2} \right]_1^b \\ &= \lim_{b \rightarrow +\infty} \left[-\frac{1}{2b^2} + \frac{1}{2} \right] \\ &= \frac{1}{2} \end{aligned}$$



⊗ စာလိပ်ကိစ္စ integral အား အကုန်ပေါ် $\int_1^{+\infty} \frac{1}{x^3} dx$ converge (အံ့ပျက်)။
 ကိန်းတန်ဖိုး $\frac{1}{2}$

Example 7.22 Compute $\int_0^{+\infty} \cos x dx$.

$$\begin{aligned} \int_0^{+\infty} \cos x dx &= \lim_{b \rightarrow +\infty} \int_0^b \cos x dx \\ &= \lim_{b \rightarrow +\infty} [\sin x]_0^b \\ &= \lim_{b \rightarrow +\infty} [\sin b - \sin 0] \\ &= \text{DNE} \end{aligned}$$



⊗ စာလိပ်ကိစ္စ integral အား အကုန်ပေါ် $\int_0^{+\infty} \sin x dx$ diverge (အံ့ပျက်)။

Example 7.23 Compute $\int_1^{\infty} \frac{\ln x}{x} dx$.

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow +\infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow +\infty} \left[\frac{(\ln x)^2}{2} \right]_1^b \\ &= \lim_{b \rightarrow +\infty} \left[\frac{(\ln b)^2}{2} - \frac{(\ln 1)^2}{2} \right] \\ &= +\infty \quad \text{diverges} \end{aligned}$$



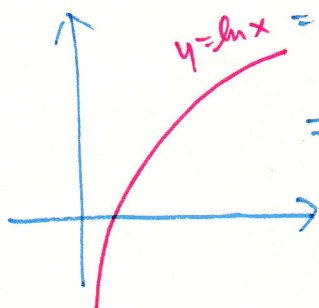
သို့သော် $\int \frac{\ln x}{x} dx$ u du

Let $u = \ln x$; $du = \frac{1}{x} dx$

$\therefore \int \frac{\ln x}{x} dx = \int u du$

$= \frac{u^2}{2} + C$

$= \frac{(\ln x)^2}{2} + C$



Example 7.24 Compute $\int_{-\infty}^1 \frac{1}{3-2x} dx$.

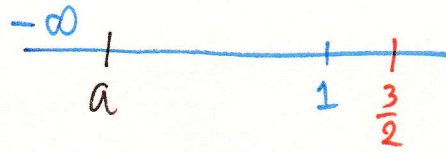
$$\int_{-\infty}^1 \frac{1}{3-2x} dx = \lim_{a \rightarrow -\infty} \int_a^1 \frac{1}{3-2x} dx$$

$$= \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} \ln|3-2x| \right]_a^1$$

$$= \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} \ln(1) + \frac{1}{2} \ln|3-2a| \right]$$

$$= +\infty \text{ diverge.}$$

$\times a \rightarrow -\infty \Rightarrow -2a \rightarrow +\infty$
 $3-2a \rightarrow +\infty$



$$\int \frac{1}{3-2x} dx$$

$$\text{Let } u = 3-2x; du = -2dx$$

$$\therefore \int \frac{1}{3-2x} dx = \int \frac{1}{u} \cdot \frac{dx}{du} du = -\frac{1}{2} \int \frac{1}{u} du$$

$$= -\frac{1}{2} \ln|u| + C$$

$$= -\frac{1}{2} \ln|3-2x| + C$$

Example 7.25 Compute $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$.

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{x}{x^2+1} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{x}{x^2+1} dx$$

$$= \lim_{a \rightarrow -\infty} \left[\frac{1}{2} \ln|a^2+1| - \frac{1}{2} \ln|a^2+1| \right] +$$

$$\lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln|b^2+1| - \frac{1}{2} \ln|1| \right]$$

$$\circledast \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} \ln|a^2+1| \right] = -\infty \text{ diverges.}$$

$$\text{Therefore, } \int_{-\infty}^{\infty} \frac{x}{x^2+1} dx \text{ diverge.}$$

* တကယ်တမ်း အကဲခတ်
လျှော့ကုန်ကုန်လျှော့
diverge
နဲ့ အနုစိတ်
diverge ကိုခတ်!

$$\int \frac{x}{x^2+1} dx$$

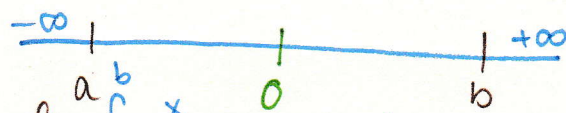
$$\text{Let } u = x^2+1; du = 2x dx$$

$$\therefore \int \frac{x}{x^2+1} dx = \int \frac{x}{u} \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|x^2+1| + C$$



7.6.2 Integrals whose integrands have infinite discontinuities:

Let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x) dx$ as the area of the region. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y -direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, k]$, where $a \leq k < b$, and then letting k approach b to fill out the area of the entire region.

