- Newton
- Leibnitz.

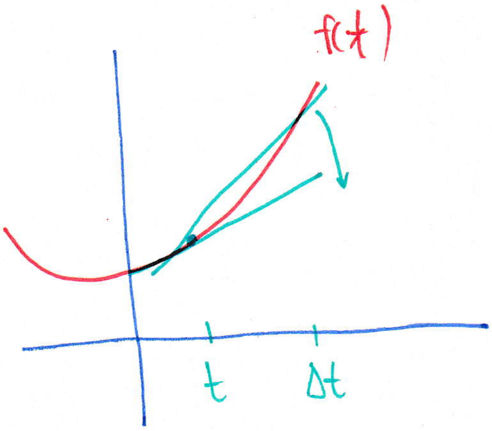
$\frac{dy}{dx}$ / $\int \cdots dx$

- $P(t)$
- $C(t)$
- $S(t)$

$\frac{\Delta C}{\Delta t}$

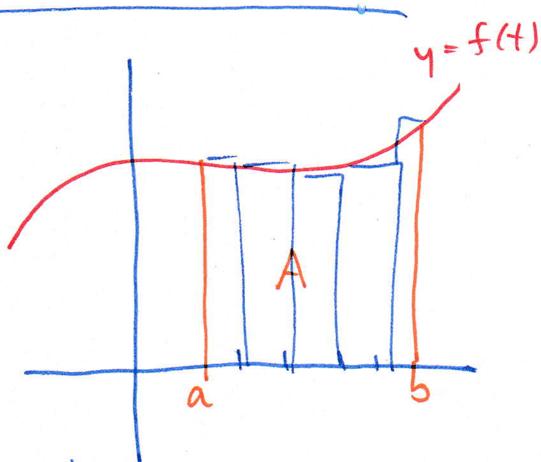
$$\frac{dy}{dx} = +5 \text{ m/s.}$$

$$\frac{dC}{dt} = -6 \text{ molar/s.}$$



$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Differential Calculus



$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

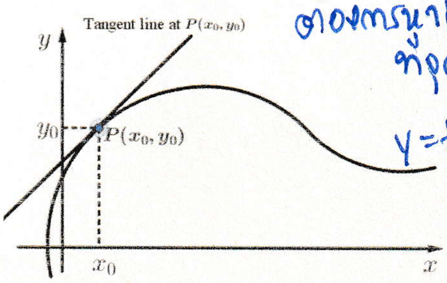
Integral Calculus

$\boxed{\text{Fundamental Theorem of Calculus}}$

1

Limits and Continuity

> **The Tangent Line Problem** Given a function f and a point $P(x_0, y_0)$ on the graph of f, find an equation of the line that is tangent to the graph of f at P. (Figure 1.1)



ต้องการหาเส้นสัมผัสกราฟ $y=f(x)$
ที่จุด (x_0, y_0)

$y=f(x)$

Figure 1.1: A picture of tangent line at point P

Example 1.1 Find an equation for the tangent line to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

Solution.

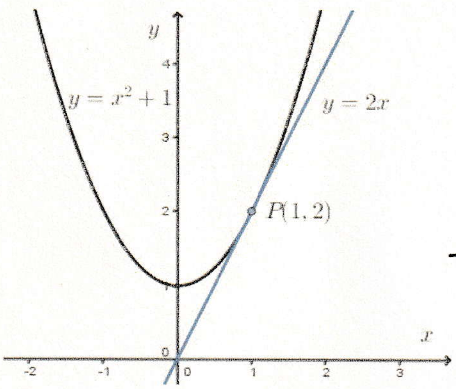
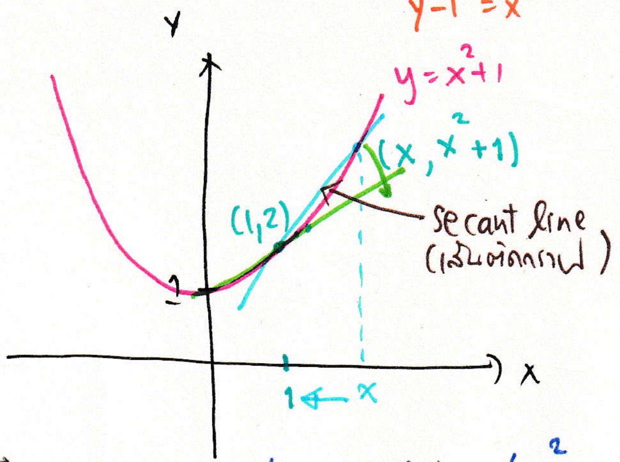
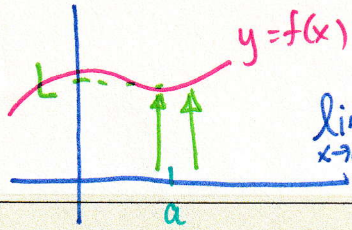


Figure 1.2: Graph of $y = x^2 + 1$



$$\text{Slope} = \frac{\Delta Y}{\Delta X} = \frac{(x^2+1) - 2}{x-1} = \frac{x^2-1}{x-1}$$

ถ้าขยับให้ x เข้าใกล้ 1
จะได้ความชันของเส้นสัมผัสที่จุด $(1,2)$



$\lim_{x \to a} f(x) = L$

1.1 Limits

> LIMITS If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write
>
> $$\lim_{x \to a} f(x) = L$$
>
> which is read "the limit of $f(x)$ as x approaches a is L", or "$f(x)$ approaches L as x approaches a".

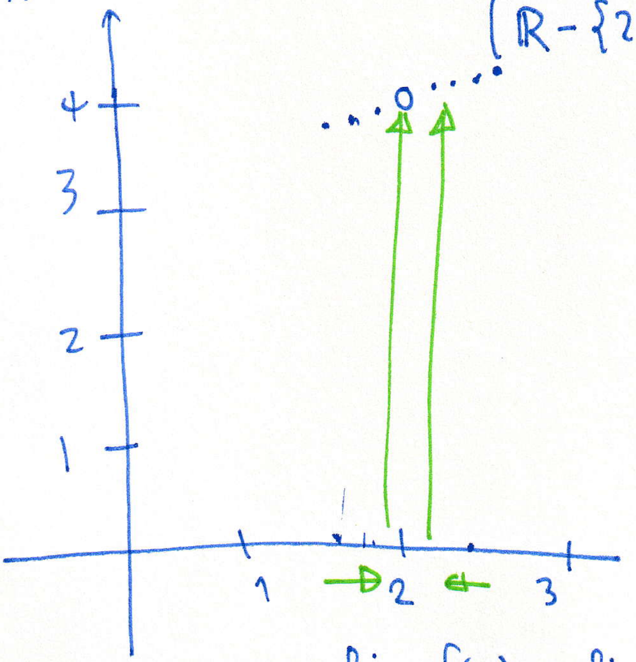
Example 1.2 Use numerical evidence to make a conjecture about the value of $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution.

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	3.9999	4.0001	4.001	4.01	4.1

$f(x) = \frac{x^2-4}{x-2}$ โดเมนของ f คือ $\{x \in \mathbb{R} \mid x \neq 2\}$

$(\mathbb{R} - \{2\})$



$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

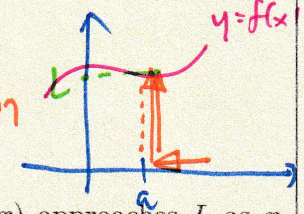
* ถ้าลิมิต มีค่า ค่าของลิมิตจะต้องเป็นตัวเลข (ที่เป็นจำนวนจริง)

1.1.1 One-sided Limits

One-sided Limits If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\boxed{\lim_{x\to a^+} f(x) = L}$$

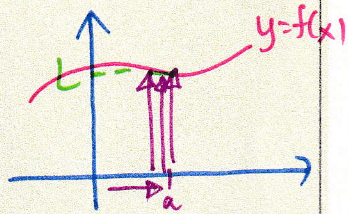
ลิมิตทางขวา



("the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right".)

and if the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\boxed{\lim_{x\to a^-} f(x) = L}$$



("the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left".)

Example 1.3 Explain why $\displaystyle\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Solution.

The Relationship between One-Sided and Two-Sided Limits The two-sided limit of a function $f(x)$ exists at $x = a$ if and only if both of the one-sided limits exist at a and have the same value; that is,

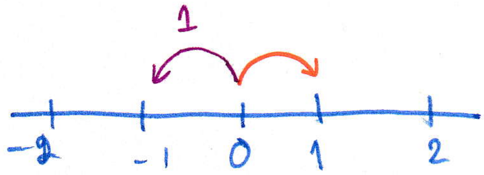
$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

ฟังก์ชันที่นิยามเป็นช่วงๆ (piecewise-defined function)

$$f(x) = \begin{cases} \_\_\_\_\_\_ \; ; & x \geq \_\_\_\_ \\ \_\_\_\_\_\_ \; ; & < x \; \_\_\_\_ \end{cases}$$

Ex $f(x) = \dfrac{|x|}{x}$ พิจารณา $\lim\limits_{x \to 0} f(x) \left(\lim\limits_{x \to 0} \dfrac{|x|}{x} \right)$

ทบทวน $f(x) = |x|$



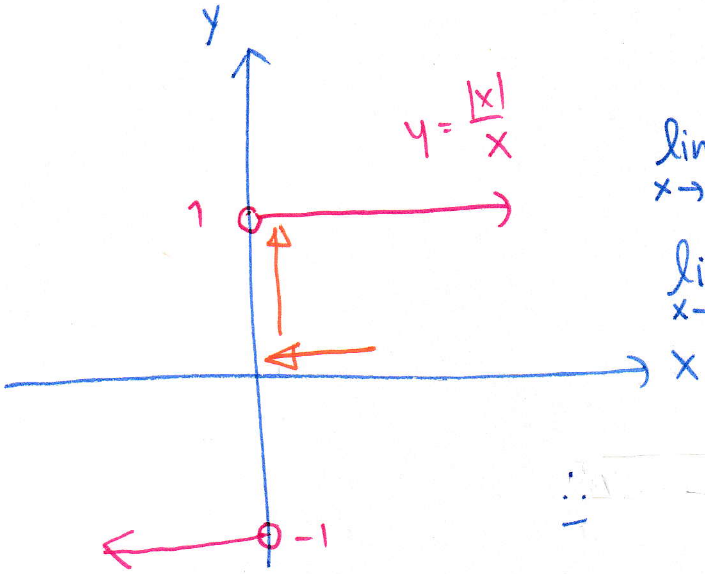
$\boxed{f(1) = 1}$

$\boxed{f(-1) = 1}$ $\Rightarrow f(-1) = -(-1) = +1$

$$f(x) = \begin{cases} \boxed{x \; ; \; x \geq 0} \\ \boxed{-x \; ; \; x < 0} \end{cases}$$

$$|x| = \begin{cases} x \; ; & x \geq 0 \\ -x \; ; & x < 0 \end{cases}$$

$$\therefore \frac{|x|}{x} = \begin{cases} \dfrac{x}{x} & ; x > 0 \\ -\dfrac{x}{x} & ; x < 0 \end{cases} \Rightarrow \frac{|x|}{x} = \begin{cases} \boxed{1 \; ; \; x > 0} \\ \boxed{-1 \; ; \; x < 0} \end{cases}$$



$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -1 = -1$$

$$\therefore \lim_{x \to 0^+} \frac{|x|}{x} = 1 \neq -1 = \lim_{x \to 0^-} \frac{|x|}{x}$$

จึงได้ว่า $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ ไม่มีค่า #

Example 1.4 For the functions in Figure 1.3 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.

does not exist

$\lim_{x\to a^+} f(x) = 1$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x) =$ DNE

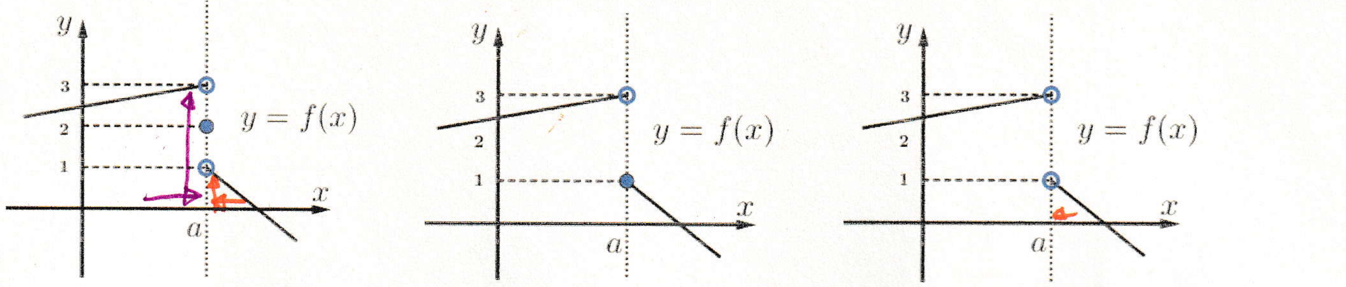
$f(a) = 2$

$\lim_{x\to a^+} f(x) = 1$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x)$ DNE

$f(a) = 1$

$\lim_{x\to a^+} f(x) = 1$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x)$ DNE

$f(a)$ ไม่นิยาม



(a) (b) (c)

Figure 1.3: A picture for Exercise 1.4

Example 1.5 For the functions in Figure 1.4 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.

$\lim_{x\to a^+} f(x) = 3$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x) = 3$

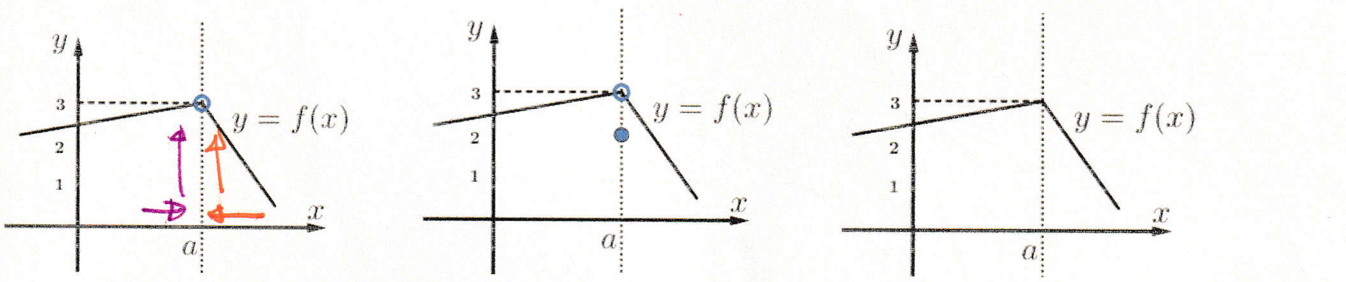
$f(a)$ ไม่นิยาม

$\lim_{x\to a^+} f(x) = 3$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x) = 3$

$f(a) = 2$

$\lim_{x\to a^+} f(x) = 3$, $\lim_{x\to a^-} f(x) = 3$ $\Big\}$ $\lim_{x\to a} f(x) = 3$

$f(a) = 3$



(a) (b) (c)

Figure 1.4: A picture for Exercise 1.5

1.1.2 Infinite Limits

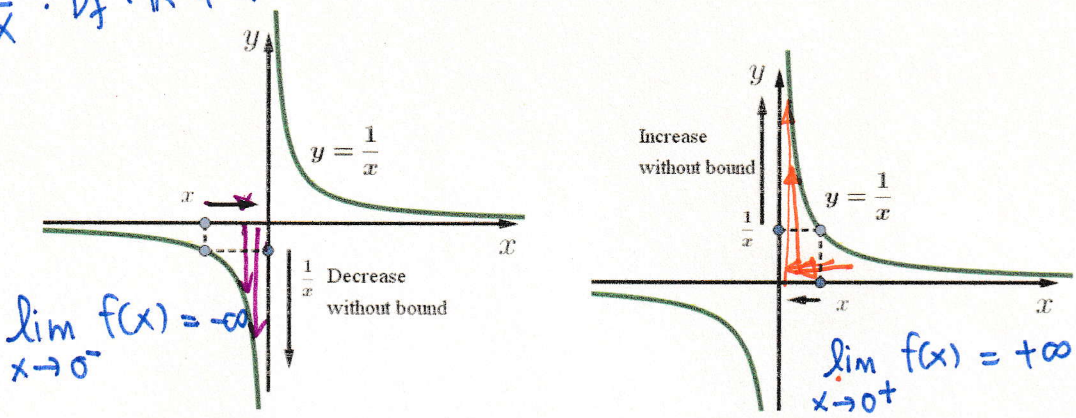
ลิมิตอนันต์ : ลิมิตของฟังก์ชันเมื่อค่า $x \to a$ แล้ว ค่าของฟังก์ชัน $f(x)$ เพิ่มขึ้นหรือ ลดลงอย่างไม่มีขอบเขต $+\infty$ $-\infty$

Sometimes the values of the function increase or decrease without bound.

x	-10	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	0
$\frac{1}{x}$	-0.1	-1	-10	-100	-1000	-10,000	$\cdots$	

$f(x) = \frac{1}{x}$: $D_f : \mathbb{R} - \{0\}$

$(xy = 1)$



$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = +\infty$

Figure 1.5: (Left) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ (Right) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

x	0	$\cdots$	0.0001	0.001	0.01	0.1	1	10
$\frac{1}{x}$		$\cdots$	10,000	1000	100	10	1	0.1

The above discussion represents the limit of $1/x$ as $x \to 0^-$ and $x \to 0^+$ graphically. These can be summarized that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

Let us next consider the limit of $1/(x-a)$ as $x \to a^-$ and $x \to a^+$.

Example 1.6 Fill in the blank and guess what are $\lim_{x \to a^-} \frac{1}{x-a}$ and $\lim_{x \to a^+} \frac{1}{x-a}$

x	$a-1$	$a-0.1$	$a-0.01$	$a-0.001$	$a-0.0001$	$\cdots$	a
$\frac{1}{x-a}$	-1	-10	-100	-1000	-10,000	$\cdots$	

x	a	$\cdots$	$a+0.0001$	$a+0.001$	$a+0.01$	$a+0.1$	$a+1$
$\frac{1}{x-a}$		$\cdots$	10,000	1000	100	10	1