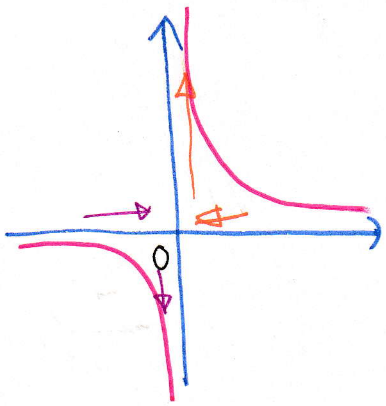
ให้ $y=f(x)$ เป็นฟังก์ชัน

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Ex $f(x) = \frac{1}{x}$



$$\therefore \lim_{x\to 0^+} f(x) = +\infty$$

$$\lim_{x\to 0^-} f(x) = -\infty$$

$$\Downarrow$$

$$\lim_{x\to 0} f(x) \text{ DNE.}$$

Infinite Limits The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty.$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

These cases can be considered as limit does not exist.

Example 1.7 For the functions in Figure 1.6 , describe the limits at $x = a$ in appropriate limit notation.

$f(x) = \frac{1}{x-a}$

$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = -\infty$

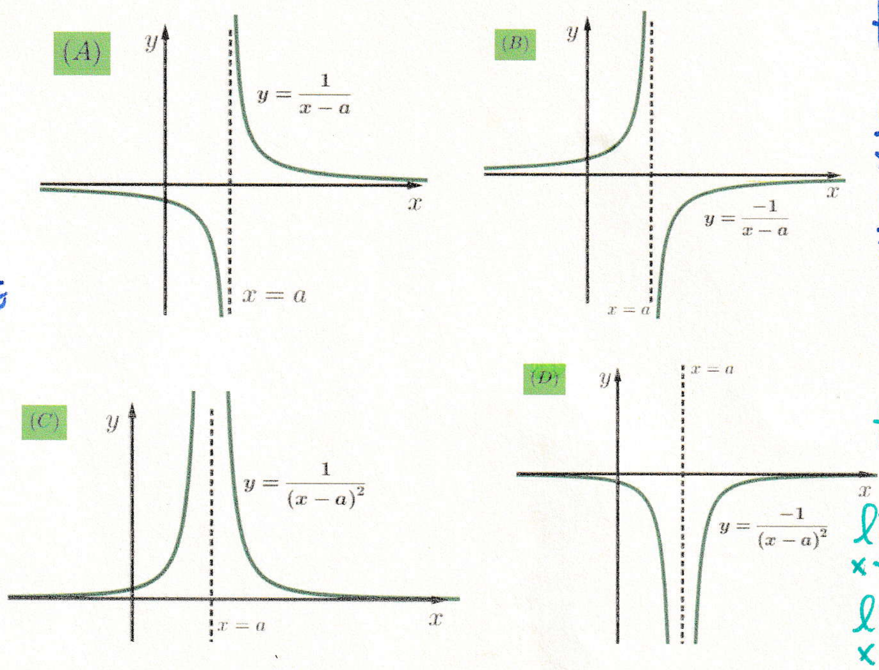
$\lim_{x \to a} f(x) = \text{DNE}$

$f(x) = -\frac{1}{x-a}$

$\lim_{x \to a^+} f(x) = -\infty$

$\lim_{x \to a^-} f(x) = +\infty$

$\lim_{x \to a} f(x) = \text{DNE}$



$f(x) = \frac{1}{(x-a)^2}$

$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = +\infty$

$\lim_{x \to a} f(x) = +\infty$

$f(x) = -\frac{1}{(x-a)^2}$

$\lim_{x \to a^+} f(x) = -\infty$

$\lim_{x \to a^-} f(x) = -\infty$

$\lim_{x \to a} f(x) = -\infty$

Figure 1.6: A picture for Exercise 1.7

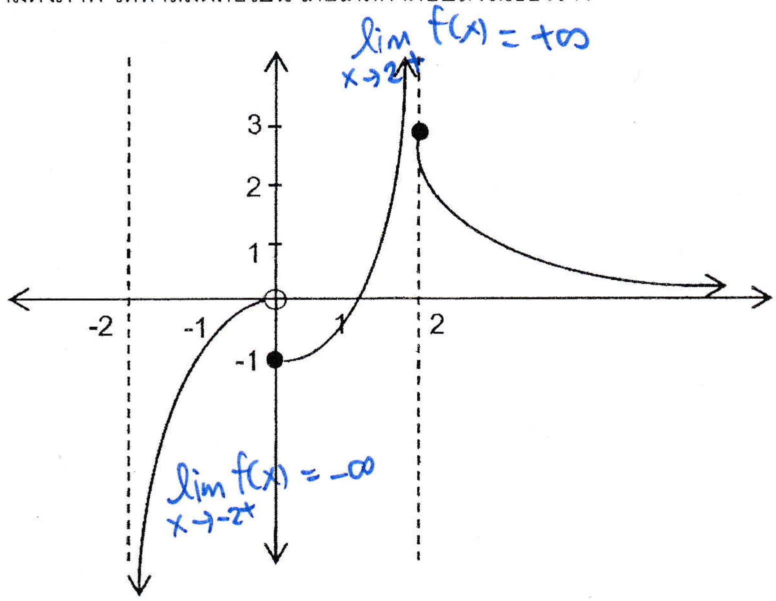
Solution.

ครั้งที่ 1: แบบทดสอบย่อย

เพื่อเช็คชื่อเข้าชั้นเรียน ประจำวันอังคารที่ 6 สิงหาคม พ.ศ.2562

ชื่อ-สกุล.. รหัสนักศึกษา............................ ลำดับที่...........

กำหนดให้ f นิยามดังภาพ ให้หาลิมิตต่อไปนี้ โดยเติมคำตอบลงในช่องว่าง



1. $\lim\limits_{x\to -2^+} f(x)$ = ...

2. $\lim\limits_{x\to 0^-} f(x)$ = ...

3. $\lim\limits_{x\to 0^+} f(x)$ = ...

4. $\lim\limits_{x\to 0} f(x)$ = ...

5. $\lim\limits_{x\to 2^-} f(x)$ = ...

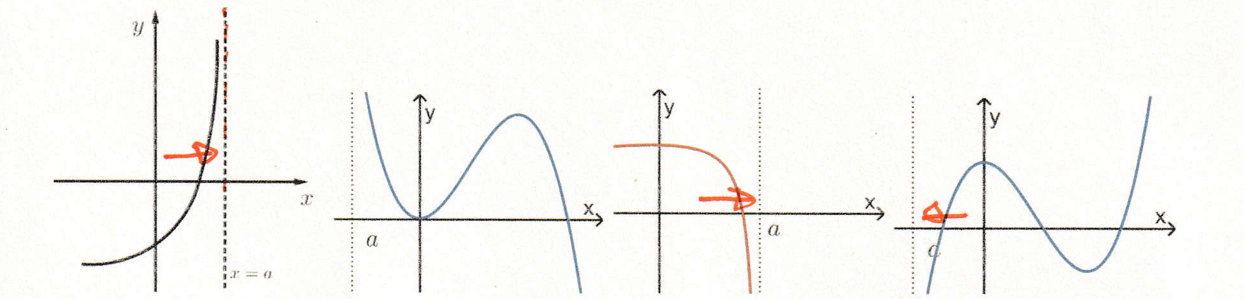
6. $\lim\limits_{x\to 2^+} f(x)$ = ...

7. The vertical asymtotes of the graph of f (เส้นกำกับแนวยืนของกราฟ f) is $X=-2, X=2$

1.1.3 Vertical Asymptotes เส้นกำกับแนวยืน/ เส้นกำกับแนวตั้ง

If the graph of $f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit, we call the line $x = a$ ***vertical asymptote*** of the curve $y = f(x)$. In other word, the line $x = a$ is ***vertical asymptote*** of $f(x)$ if $\lim_{x\to a} f(x) \to \pm\infty$.

Figure 1.7 illustrates geometrically what happen when any of the following situations occur:



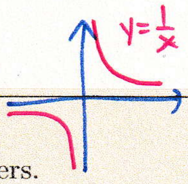
$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

Figure 1.7: Examples of vertical asymptotes

จะกล่าวว่า $x=a$ เป็นเส้นกำกับแนวยืน ก็ต่อเมื่อ $\lim f(x) = +\infty / -\infty$
(VA)
$x\to a^+$
$x\to a^-$
$x\to a$

1.2 Limits

1.2.1 Basic Limits



THEOREM 1.1 Let a and k be real numbers.

(a) $\lim_{x\to a} k = k$ (b) $\lim_{x\to a} x = a$ (c) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (d) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

Example 1.8 If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $\lim_{x\to a} k = k$ for all value of a. For example,

Solution.

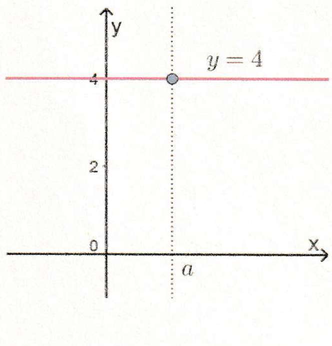


Figure 1.8: Graph of $f(x) = 4$

$\lim_{x\to -5} 4 =$ 4

$\lim_{x\to 0} 4 =$ 4

$\lim_{x\to 15} 4 =$ 4

Example 1.9 If $f(x) = x$, then the values of $f(x)$ always equals to x varies, which explains why $\lim_{x\to a} x = a$ for all value of a. For example,

Solution.

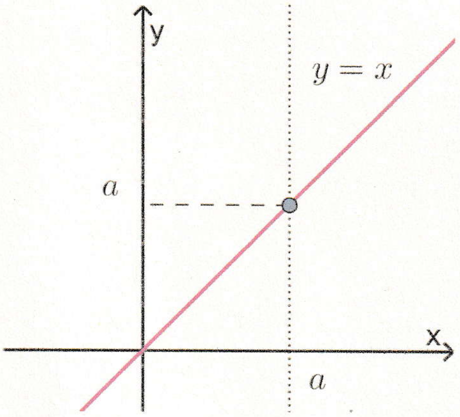


Figure 1.9: Graph of $f(x) = x$

$\lim_{x\to 0} x = 0$

$\lim_{x\to 3} x = 3$

$\lim_{x\to -100} x = -100$

$\lim_{x\to \pi} x = \pi$

> **THEOREM 1.2** Let a be a real number, and suppose that
>
> $$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} g(x) = L_2.$$
>
> That is, the limits exist and have values L_1 and L_2, respectively, Then:
>
> (a) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$
>
> (b) $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$
>
> (c) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$
>
> (d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, provided $L_2 \neq 0$
>
> (e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if n is even.
>
> Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

1.2.2 Limits of Polynomials and Rational Functions as $x \to a$

Example 1.10 Find $\lim_{x\to 2}(x^3 - 3x + 5)$.

$p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$

Solution. $\lim_{x\to 2}(x^3-3x+5) = \lim_{x\to 2} x^3 + \lim_{x\to 2}(-3x) + \lim_{x\to 2} 5$

$= \lim_{x\to 2} x^3 - 3\lim_{x\to 2} x + \lim_{x\to 2} 5$

$= (2)^3 - 3(2) + 5$

$= 8 - 6 + 5 = 7$

$p(x) = x^3 - 3x + 5$

$p(2) = (2)^3 - 3(2) + 5 = 7$

THEOREM 1.3 For any polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

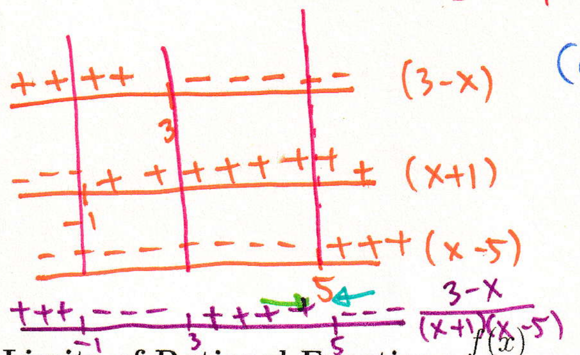
Rational function (ฟังก์ชันตรรกยะ) $f(x)=\dfrac{p(x)}{q(x)}$; $p(x), q(x)$ เป็นพหุนาม.
$(p(a)\neq 0$ แต่ $q(a)=0)$

Example 1.11 Find

(a) $\displaystyle\lim_{x\to 5^+}\frac{3-x}{(x-5)(x+1)}$ (b) $\displaystyle\lim_{x\to 5^-}\frac{3-x}{(x-5)(x+1)}$ (c) $\displaystyle\lim_{x\to 5}\frac{3-x}{(x-5)(x+1)}$

Solution. (a) $\displaystyle\lim_{\boxed{x\to 5^+}}\frac{3-x^{-}}{\underset{+}{(x-5)}\underset{+}{(x+1)}}=-\infty$

(b) $\displaystyle\lim_{\boxed{x\to 5^-}}\frac{3-x^{-}}{\underset{-}{(x-5)}\underset{+}{(x+1)}}=+\infty$



(c) $\displaystyle\lim_{x\to 5}\frac{3-x}{(x-5)(x+1)}$ DNE.

Limits of Rational Functions $\dfrac{f(x)}{g(x)}$ **as** $x\to a$ **when** $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ $\boxed{\frac{0}{0}}$ indeterminate form

Example 1.12 Find

(a) $\displaystyle\lim_{x\to 5}\frac{x^2-10x+25}{x-5}$ (b) $\displaystyle\lim_{x\to 2}\frac{3x-6}{x^2-x-2}$ (c) $\displaystyle\lim_{x\to 3}\frac{x^2-5x+6}{x^2-6x+9}$.

Solution. (a) $\displaystyle\lim_{x\to 5}\frac{x^2-10x+25}{x-5}$ $p(x)=x^2-10x+25 \Rightarrow p(5)=5^2-10(5)+25=0$

$q(x)=x-5 \Rightarrow q(5)=5-5=0$

$\displaystyle=\lim_{x\to 5}\frac{\cancel{(x-5)}(x-5)}{\cancel{x-5}}$

$\boxed{f(x)=\dfrac{x^2-4}{x-2}}$

$\displaystyle=\lim_{x\to 5} x-5$

$=0$

(b) $\displaystyle\lim_{x\to 2}\frac{3x-6^{\,0}}{x^2-x-2_{\,0}}=\lim_{x\to 2}\frac{3\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}=\lim_{x\to 2}\frac{3}{x+1}=\frac{3}{3}=1$

(c) $\displaystyle\lim_{x\to 3}\frac{x^2-5x+6^{\,0}}{x^2-6x+9^{\,0}}=\lim_{x\to 3}\frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x-3)}=\boxed{\lim_{x\to 3}}\frac{x-2}{x-3}$

แยกคิดลิมิตสองข้าง $\displaystyle\lim_{x\to 3^-}\frac{x-2^{\oplus}}{x-3_{\ominus}}=-\infty$

$\displaystyle\lim_{x\to 3^+}\frac{x-2^{\oplus}}{x-3_{\oplus}}=+\infty$

$\displaystyle\therefore \lim_{x\to 3}\frac{x-2}{x-3}$ ไม่มี

THEOREM 1.4 Let
$$f(x) = \frac{p(x)}{q(x)}$$
be a rational function, and let a be any real number.

(a) If $q(a) \neq 0$, then $\lim\limits_{x \to a} f(x) = f(a)$.

(b) If $q(a) = 0$ but $p(a) \neq 0$, then $\lim\limits_{x \to a} f(x)$ does not exist. (ระบุได้อีกหน่อยว่าเป็น $+\infty$ / $-\infty$ / บอกไม่ได้เลย)

Limits Involving Radicals

Example 1.13 Find $\lim\limits_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$.

conjugate ของ $\sqrt{x}-2$ (สังยุค)

Solution.

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} \times \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x \to 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x})^2 - 2^2}$$

$$= \lim_{x \to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}}$$

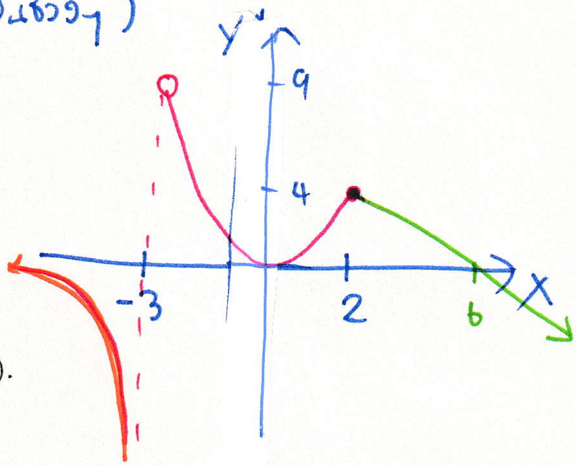
$$= \lim_{x \to 4} \sqrt{x}+2 = 4$$

หรือ
$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}}$$

$$= \lim_{x \to 4} \sqrt{x}+2$$

$$= 4$$

Limits of Piecewise-Defined Functions (ฟังก์ชันที่นิยามเป็นช่วงๆ)



Example 1.14 Let

$$f(x) = \begin{cases} \boxed{\dfrac{2}{x+3}, \quad x < -3} \\ \boxed{x^2, \quad -3 < x \le 2} \\ \boxed{6 - x, \quad x > 2} \end{cases}$$

Find (a) $\lim_{x \to -3} f(x)$ (b) $\lim_{x \to 0} f(x)$ (c) $\lim_{x \to 2} f(x)$.

Solution.

(a) $\lim_{x \to -3} f(x) \longrightarrow \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \dfrac{2}{x+3} = -\infty$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} x^2 = 9$

$\therefore \lim_{x \to -3} f(x) = \text{DNE}$

(b) $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$

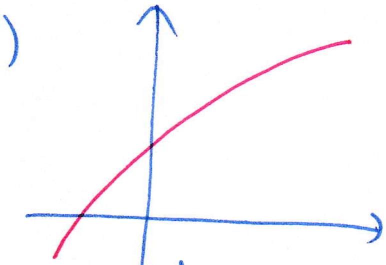
(c) $\lim_{x \to 2} f(x) \longrightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 6 - x = 4$

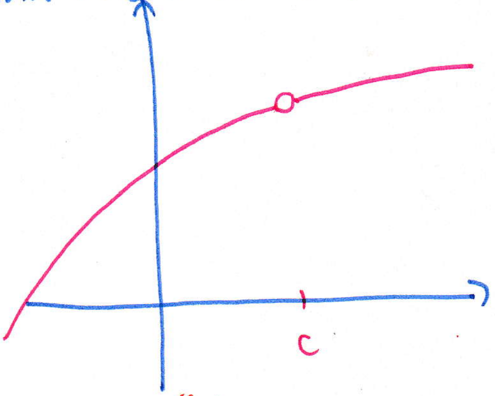
$\therefore \lim_{x \to 2} f(x) = 4$

ความต่อเนื่องของฟังก์ชัน (continuity of function)

$y=f(x)$ is discontiuous at $x=c$.

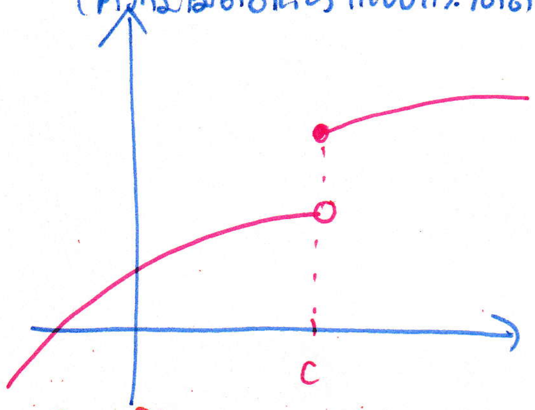


$\boxed{1}$ Removable discontinuity
(ความไม่ต่อเนื่องแบบขจัด)



$f(c)$ ไม่นิยาม

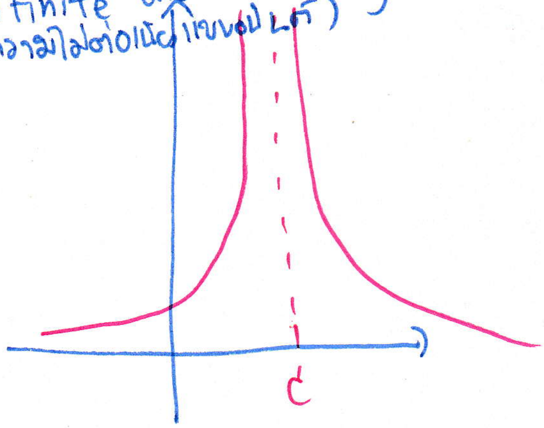
$\boxed{2}$ jump discontinuity
(ความไม่ต่อเนื่องแบบกระโดด)



$\boxed{1}$ $\lim_{x \to c} f(x)$ DNE (ลิมิตซ้าย $\neq$ ลิมิตขวา)

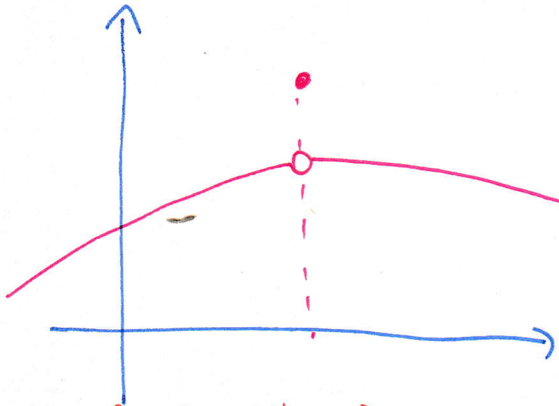
$\boxed{2}$ $f(c)$ จะนิยามหรือไม่ก็ได้

$\boxed{3}$ infinite discontinuity
(ความไม่ต่อเนื่องแบบอนันต์)



$\lim_{x \to c} f(x) = +\infty / -\infty /$ ไม่มี

$\boxed{4}$ Removable discontinuity.



$\boxed{1}$ $\lim_{x \to c} f(x)$ มี, $f(c)$ มี

$\boxed{2}$ $\lim_{x \to c} f(x) \neq f(c)$

1.3 Continuity

DEFINITION A function f is said to be *continuous at* $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim\limits_{x\to c} f(x)$ exists.
3. $\lim\limits_{x\to c} f(x) = f(c)$.

Example 1.15 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \boxed{\dfrac{x^2-4}{x-2}, \quad x \neq 2} \\ \boxed{2, \quad x = 2,} \end{cases} \qquad h(x) = \begin{cases} \boxed{\dfrac{x^2-4}{x-2}, \quad x \neq 2} \\ \boxed{4, \quad x = 2.} \end{cases}$$

Solution. ① $f(x) = \dfrac{x^2-4}{x-2}$ $\quad (D_f = \mathbb{R} - \{2\})$

$f(2)$ ไม่นิยาม

$\therefore$ f ไม่ต่อเนื่องที่ $x = 2$

② $g(2) = 2$

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$$

$$\lim_{x\to 2} g(x) = 4 \neq g(2)$$

g ไม่ต่อเนื่องที่ $x = 2$

③ $h(x) = \begin{cases} \boxed{\dfrac{x^2-4}{x-2} \; ; \; x \neq 2} \\ \boxed{4 \quad ; x = 2} \end{cases}$

$h(2) = 4$

$$\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

$$\therefore \lim_{x\to 2} h(x) = 4 = h(2)$$

$\therefore$ h ต่อเนื่องที่ $x = 2$.