ทบทวน ให้ f เป็นฟังก์ชัน อนุพันธ์ของฟังก์ชันที่ x

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

อนุพันธ์ที่ $x_0 \rightarrow f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

หรือ $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ ⊛ อนุพันธ์มีค่า / $\overset{\text{differentiable}}{\text{หาอนุพันธ์ได้}}$ ที่ $x = x_0$

ถ้า ลิมิต ⊛ มีค่า

$$\Updownarrow$$

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

Ex 2.6 พิจารณา $f(x) = |x|$ จงหาว่า f หาอนุพันธ์ได้ที่ $x=0$ หรือไม่

Sol^n $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ มีไหม?

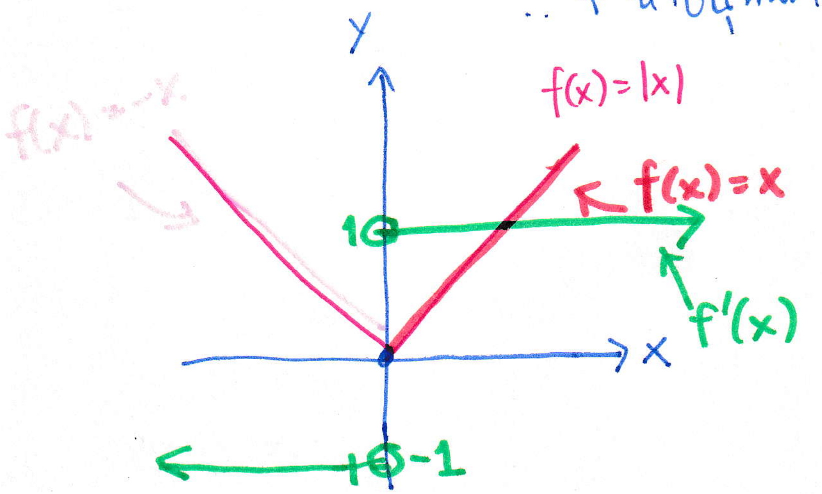
พิจารณา $f(x) = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases} \Big\}$ @ $x_0 = 0$

พิจารณาลิมิตขวา $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(0+h) - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-(0+h) - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

เนื่องจาก $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1 \neq -1 = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$\therefore$ f หาอนุพันธ์ไม่ได้ที่ $x = 0$



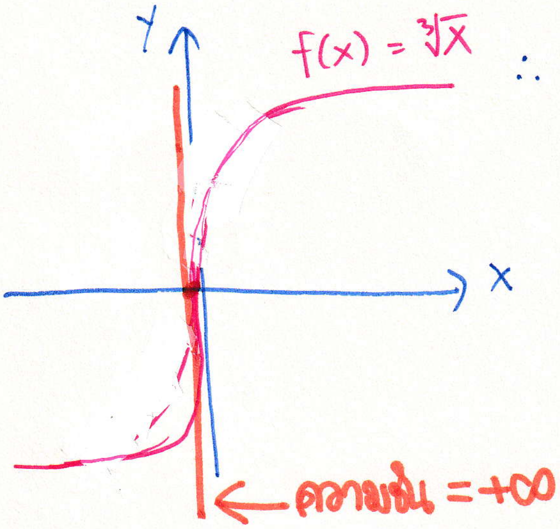
$$\boxed{f'(x) = \begin{cases} 1 & ; x > 0 \\ -1 & ; x < 0 \end{cases}}$$

* $f'(0)$ ไม่มีค่า

Example 2.7 (a) Prove that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$ by considering the limit.

(b) Find a formula for $f'(x)$. (c) Does it has any vertical tangent ?

Solution. $f'(0)$ มีไหม : $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$?



" เส้นสัมผัสแนวชัน "
(vertical tangent)

$$\therefore \lim_{h\to 0} \frac{\sqrt[3]{0+h}-\sqrt[3]{0}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h}$$

$$= \lim_{h\to 0} \frac{h^{1/3}}{h^1} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$$

$$\therefore \lim_{h\to 0} \frac{\sqrt[3]{0+h}-\sqrt[3]{0}}{h} = +\infty \quad (\text{D.N.E})$$

$\therefore f(x) = \sqrt[3]{x}$ หาอนุพันธ์ไม่ได้ที่ $x = 0$

สำหรับ x ใดๆ ที่ $\underline{\underline{x \neq 0}}$; $\lim_{h\to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$ $[(\sqrt[3]{x+h})^2 +$

$\downarrow$

The Relationship between Differentiability and Continuity

ถ้า f หาอนุพันธ์ได้ที่ ~~x_0~~ แล้ว f จะต่อเนื่องที่ $x = x_0$.

$\overset{p}{}$ $\overset{q}{}$

THEOREM 2.1 If a function f is differentiable at x_0, then f is continuous at x_0.

↳ เป็นข้อความเดียวกับ:
ถ้า f ไม่ต่อเนื่องที่ $x=x_0$ แล้ว f หาอนุพันธ์ไม่ได้ที่ $x=x_0$

$\boxed{p \Rightarrow q \equiv \sim q \Rightarrow \sim p}$

Note that the converse of the theorem above is false, since continuous does not imply differentiable.

2.2.3 Other Derivative Notations

When the independent variable is x, the derivative is commonly denoted by

$$f'(x) \quad \text{or} \quad \frac{d}{dx}[f(x)] \quad \text{or} \quad D_x[f(x)]$$

↙ ใช้ใน Cal 2

In the case where the dependent variable is $\boxed{y = f(x)}$, the derivative is also denoted by

$$y' \quad \text{or} \quad y'(x) \quad \text{or} \quad \frac{dy}{dx}$$

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) \quad \text{or} \quad \frac{d}{dx}[f(x)]\Big|_{x=x_0} \quad \text{or} \quad D_x[f(x)]|_{x=x_0} \quad \text{or} \quad y'(x_0) \quad \text{or} \quad \boxed{\frac{dy}{dx}\Big|_{x=x_0}}$$

Evaluating symbol.

สำหรับ $x \neq 0$

$$\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} = \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\cdot\frac{\left[(\sqrt[3]{x+h})^2+\sqrt[3]{x(x+h)}+(\sqrt[3]{x})^2\right]}{(\sqrt[3]{x+h})^3+\sqrt[3]{x(x+h)}+(\sqrt[3]{x})^2}$$

เอกลักษณ์ $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$= \lim_{h\to 0}\frac{(\sqrt[3]{x+h})^3-(\sqrt[3]{x})^3}{h\left[(\sqrt[3]{x+h})^2+\sqrt[3]{x(x+h)}+(\sqrt[3]{x})^2\right]}$$

$$= \lim_{h\to 0}\frac{\cancel{x}+\cancel{h}^{1}-\cancel{x}}{\cancel{h}\left[(\sqrt[3]{x+h})^2+\sqrt[3]{x(x+h)}+(\sqrt[3]{x})^2\right]}$$

$$= \lim_{h\to 0}\frac{1}{\left[(\sqrt[3]{x+h})^2+\sqrt[3]{x(x+h)}+(\sqrt[3]{x})^2\right]}$$

$$= \frac{1}{3(\sqrt[3]{x})^2}$$

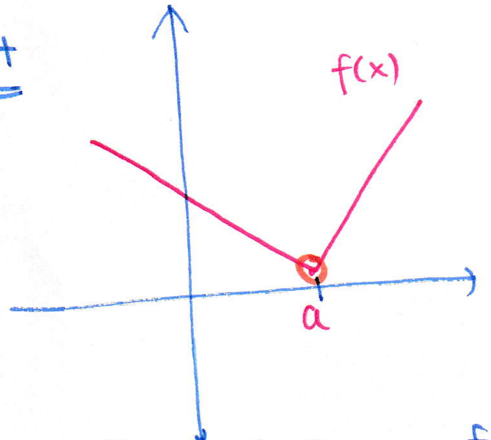
$$\therefore \boxed{f'(x)=\frac{1}{3(\sqrt[3]{x})^2} \quad \text{เมื่อ } x \neq 0}$$

\#

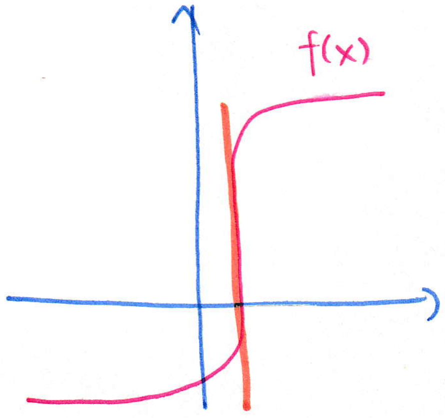
[เมื่อ $x=0$ หาอนุพันธ์ไม่ได้] ☺

ฟังก์ชันที่หาอนุพันธ์ไม่ได้ที่ $x=a$

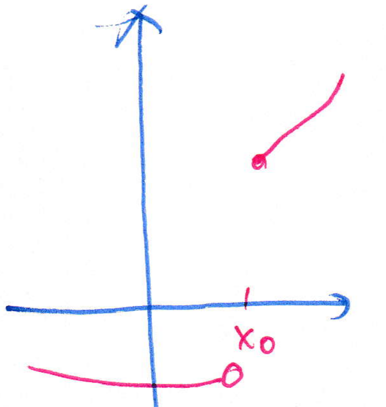
f cont



f หักงอที่ $x=a$



f มีเส้นสัมผัสแนวตั้งที่ $x=a$



f ไม่ต่อเนื่องที่ $x=a$

Thm 2.1 ถ้า f หาอนุพันธ์ได้ที่ $x=x_0$ แล้ว f ต่อเนื่องที่ $x=x_0$

ขากลับ ถ้า f ต่อเนื่องที่ $x=a$ แล้ว f หาอนุพันธ์ได้ที่ $x=a$ ไม่จริง!

Ex $f(x)=|x|$ / $f(x)=\sqrt[3]{x}$

2.3 Basic Differentiation Formulas

Recap Henceforth, we will use the following notation for the derivative of a function $y = f(x)$.

- $f'(x)$
- $\frac{d}{dx}[f(x)]$
- $\frac{dy}{dx}$
- y'

Derivative of a Constant

Theorem 2.2 (The Constant rule) Let c be a constant and $f(x) = c$ be a constant function, then

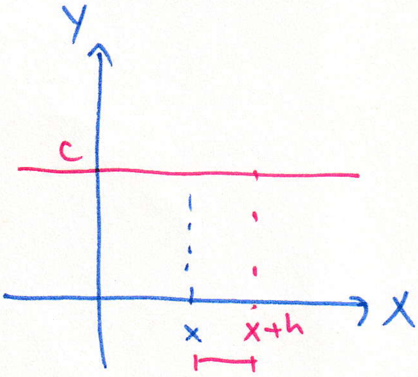
$$\frac{d}{dx}[c] = 0$$

Proof ให้ $y = f(x) = c$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0$$



Example 2.8

$\frac{d}{dx}[5] = 0$ $\quad \frac{d}{dx}[-7] = 0$ $\quad \frac{d}{dx}[5^2] = 0$

$\frac{d}{dx}[\pi] = 0$ $\quad \frac{d}{dx}[\sqrt{2}] = 0$ $\quad \frac{d}{dx}[\pi^2] = 0$

$\frac{d}{dx}[e^{\pi}] = 0$; $\frac{d}{dx}[\ln \pi] = 0$

$\frac{d}{dx}[a^x]$ (Ex $\frac{d}{dx}(2^x)$), $\frac{d}{dx}(x^x)$

Derivatives of Power Functions

* ฐานเป็นตัวแปร, เลขชี้กำลังเป็นตัวเลข

Theorem 2.3 (The Power Rule) If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\boxed{\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}}$$

Example 2.9

$\frac{d}{dx}[x^5] = 5x^4$ $\quad$ $\frac{d}{dx}[x] = 1 \cdot x^{1-1} = 1$ $\quad$ $\frac{d}{dx}[x^{12}] = 12x^{11}$

$\frac{d}{dx}[x^{-5}] = -5x^{-5-1} = -5x^{-6}$ $\quad$ $\frac{d}{dx}[\frac{1}{x^5}] = \frac{d}{dx}(x^{-5}) = -5x^{-6}$ $\quad$ $\frac{d}{dx}[x^{\sqrt{2}}] = \sqrt{2}x^{(\sqrt{2}-1)}$

$\frac{d}{dx}[x^{\frac{2}{5}}] = \frac{2}{5}x^{\frac{2}{5}-1} = \frac{2}{5}x^{-3/5}$

$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}[\frac{1}{\sqrt{x}}] = \frac{d}{dx}[\frac{1}{x^{1/2}}] = \frac{d}{dx}[x^{-1/2}] = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2(\sqrt{x})^3}$

Derivative of a Constant Times a Function

Theorem 2.4 (The Constant Multiple Rule) If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example 2.10 $\frac{d}{dx}(4x^8) = 4\frac{d}{dx}(x^8) = (4)(8)x^7 = 32x^7$

$\frac{d}{dx}(-x^{12}) = -\frac{d}{dx}(x^{12}) = -12x^{11}$

$\frac{d}{dx}\left(\frac{x}{\pi}\right) = \frac{1}{\pi}\frac{d}{dx}(x) = \frac{1}{\pi}$

Derivatives of Sums and Differences

Theorem 2.5 (The Sum and Difference Rules) If f and g are differentiable at x, then so are $f+g$ and $f-g$ and

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example 2.11 $\frac{d}{dx}[x^8 + x^2] = \frac{d(x^8)}{dx} + \frac{d}{dx}(x^2) = 8x^7 + 2x$

$\frac{d}{dx}[6x^{11} - 9] = 66x^{10} - 0 = 66x^{10}$

$\frac{d}{dx}\left(\frac{1}{x} - x^{-9}\right) = -x^{-2} - (-9)x^{-10} = -x^{-2} + 9x^{-10} = -\frac{1}{x^2} + \frac{9}{x^{10}}$

$\frac{d}{dx}\left(\frac{\sqrt{x} - 3x}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{\sqrt{x}}{\sqrt{x}} - \frac{3x}{\sqrt{x}}\right) = \frac{d}{dx}\left(1 - 3\sqrt{x}\right) = -\frac{3}{2\sqrt{x}}$ $\left(\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right)$

เส้นสัมผัสแนวนอน

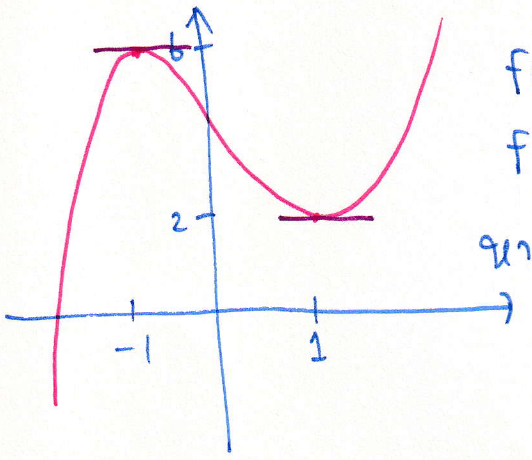
Example 2.12 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

(slope = 0

$\Downarrow$

หา x ที่ $\boxed{f'(x) = 0}$

Solution.



$f(x) = x^3 - 3x + 4$

$f'(x) = 3x^2 - 3$

หา x ที่ทำให้ $f'(x) = 0$:

Set $f'(x) = 3x^2 - 3 = 0$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$\therefore x = -1, 1$

$\therefore$ ที่ $x = -1, 1$ ทำให้มีเส้นสัมผัส ผสแนวนอน.
(horizontal tangent)

2.4 Product and Quotient Rules

2.4.1 Derivative of a Product

Theorem 2.6 (The Product Rule) If f and g are differentiable at x, then so is the product $f \cdot g$, and

$$\overset{\text{น}\cdot\text{ล}}{\frac{d}{dx}[f(x)g(x)]} = \overset{\text{น}(\text{ล})' \,+\, \text{ล}(\text{น})'}{f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]}$$

Example 2.15 1. Let $y = \underline{\overset{\text{น}}{(4x^2-1)}}\,\underline{\overset{\text{ล}}{(7x^3+x)}}$. Find dy/dx.

Solution.

$y = (4x^2-1)(7x^3+x)$

$= 28x^5 - 3x^3 - x$

$\boxed{y' = 140x^4 - 9x^2 - 1}$

$\frac{dy}{dx} = (4x^2-1)\frac{d}{dx}(7x^3+x) + (7x^3+x)\frac{d}{dx}(4x^2-1)$

$\therefore \boxed{\frac{dy}{dx} = (4x^2-1)(21x^2+1) + (7x^3+x)(8x)}$

⊛ ใส่วงเล็บทุกครั้ง เสมอ

2. Let $f(x) = \underline{\overset{\text{น}}{(3x+1)}}\,\underline{\overset{\text{ล}}{(2x^2+5)}}$. Find $f'(0)$.

Solution.

$f'(x) = (3x+1)\frac{d}{dx}(2x^2+5) + (2x^2+5)\frac{d}{dx}(3x+1)$

$\therefore \boxed{f'(x) = (3x+1)(4x) + (2x^2+5)(3)}$

$\therefore f'(0) = (3\cdot 0\cancel{+1})(4\cdot 0) + (2\cdot 0+5)(3) = 15$

2.4.2 Derivative of a Quotient

THEOREM 2.7 (THE QUOTIENT RULE) If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{\text{ล}\cdot(\text{ข})' - \text{ข}(\text{ล})'}{\text{ล}^2}$$

$$\frac{d}{dx}\left[\frac{\overset{\text{ข}}{f(x)}}{\underset{\text{ล}}{g(x)}}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 2.16 Find dy/dx for y.

1. $y = \dfrac{\boxed{x^2 - 1}}{\boxed{x^4 + 1}}$

Solution.

$$\frac{dy}{dx} = \frac{(x^4+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$$

$$= \frac{(x^4+1)2x - (x^2-1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{-2x^5+4x^3+2x}{(x^4+1)^2} = \frac{-2x(x^4-2x^2-1)}{(x^4+1)^2}$$

2. $y = \dfrac{\boxed{2x + 1}}{\boxed{\sqrt{x}}}$

Solution.

$$\frac{dy}{dx} = \frac{\sqrt{x}\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} - (2x+1)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{2\sqrt{x}}{x} - \frac{(2x+1)}{2x\sqrt{x}}$$

2.3.1 Higher Derivatives

The derivative f' of a function f is itself a function and hence may have its own derivative. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as we have differentiability, we can continue the process of differentiating to obtain the third, the fourth, the fifth, and even higher derivatives of f. These successive derivatives are denoted by

the first derivative	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$
the second derivative	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$
the third derivative	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$
the fourth derivative	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A general nth order derivative	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$

Example 2.13 If $f(x) = 6x^4 - 5x^3 + x^2 - 7x + 8$,then

$f'(x) = 24x^3 - 15x^2 + 2x - 7$

$f''(x) = 72x^2 - 30x + 2$

$f'''(x) = 144x - 30$

$f^{(\underline{\underline{4}})}(x) = 144$

$f^{(5)}(x) = 0$

$\vdots$

$f^{(500)}(x) = 0$

Example 2.14 Let $y = \dfrac{x+1}{x}$. Find $y^{(111)}$.

Solution.

$$
\left.
\begin{aligned}
y &= 1 + \frac{1}{x} = 1 + x^{-1} \\
y' &= (-1)x^{-2} = -x^{-2} \\
y'' &= (-1)(-2)x^{-3} = 2x^{-3} \\
y''' &= (-1)(-2)(-3)x^{-4} = -6x^{-4} \\
y^{(4)} &= (-1)(-2)(-3)(-4)x^{-5} = 24x^{-5} \\
&= (-1)^4(1\cdot 2\cdot 3\cdot 4)x^{-4-1} \\
&\vdots \\
y^{(n)} &= (-1)^n(\underbrace{1\cdot 2\cdot 3\cdot\ldots\cdot n}_{n!})x^{-n-1}
\end{aligned}
\right\}
$$

กระบวนการ
หา
pattern.

$$\boxed{y^{(n)} = (-1)^n n!\, x^{-n-1}}$$

$$\therefore\ y^{(111)} = (-1)^{111}\cdot(111!)\,x^{-111-1}$$

$$\boxed{y^{(111)} = (-1)^{111}\cdot 111!\; x^{-112}}$$