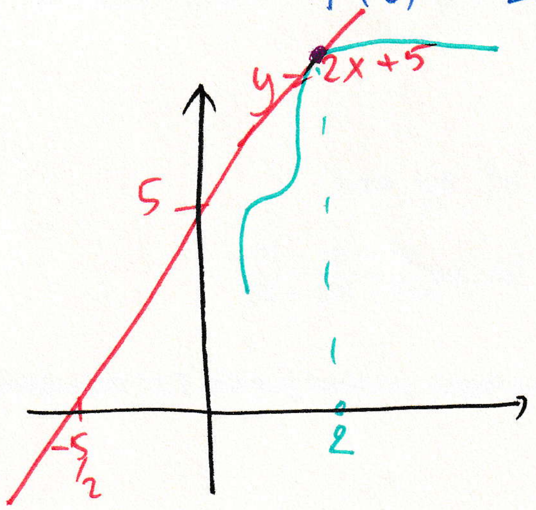
Quiz เก่า : ให้ $y = 2x+5$ เป็นเส้นสัมผัสกราฟ $y = f(x)$ ที่ $x = 2$

แล้ว $f(2) = \underline{\quad 9 \quad}$

$f'(2) = \underline{\quad 2 \quad}$



ทบทวน

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

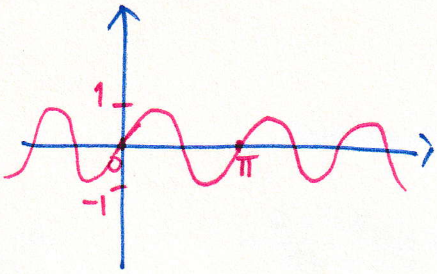
$$\boxed{\lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h}}$$

ให้ $y = f(x)$ เป็นฟังก์ชัน

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx} cf(x) = c\frac{d}{dx} f(x)$
3. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
4. $\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx} g(x) + g(x)\frac{d}{dx} f(x)$
5. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$

อนุพันธ์ของฟังก์ชันตรีโกณมิติ

$y = \sin x \qquad [f(x) = \sin x]$



$$\therefore \frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin(x+h) = \sin x \cos h + \cos x \sin h$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

(Squeeze Theorem)

$$= \lim_{h\to 0} \frac{\sin x[\cos h - 1] + \cos x \sin h}{h}$$

$$\boxed{\begin{array}{l} \lim_{h\to 0} \frac{\sin h}{h} = 1 \\ \lim_{h\to 0} \frac{1-\cos h}{h} = 0 \end{array}}$$

$$= \lim_{h\to 0} \left[\sin x \left(\cancel{\frac{\cos h - 1}{h}} \right)^{0} + \cos x \cdot \frac{\sin h}{h} \right]$$

$$= \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x$$

$f(x) = \cos x \qquad \cos(x+h) = \cos x \cos h - \sin x \sin h$

$$\frac{d}{dx}(\cos x) = -\sin x$$

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1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$

$\boxed{y = (\cos)(x)}$

~~$y' = (\cos)(x)$~~

$y' = \cos \frac{d}{dx}(x) + x \frac{d}{dx}(\cos$

$= -\sin + x \sin.$

ห้ามทำ

2.5 Derivatives of Trigonometric Functions

Before we actually get into the derivatives of the trigonometric functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos h}{h}=0.$$

Since $\sin(x+h)=\sin x\cos h+\cos x\sin h$. We have

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\left(\sin x\cos h+\cos x\sin h\right)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\left(\cos h-1\right)-\cos x\sin h}{h}\\
&= \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h-1}{h}\right)+\lim_{h\to 0}\left(\cos x\cdot\frac{\sin h}{h}\right)\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} \quad =\sin x\cdot 0+\cos x\cdot 1=\cos x
\end{aligned}$$

Thus, we have $\dfrac{d}{dx}\sin x=\cos x$.

Similaryly, using $\cos(x+h)=\cos x\cos h-\sin x\sin h$, we have

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\left(\cos x\cos h-\sin x\sin h\right)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\left(\cos h-1\right)-\sin x\sin h}{h}\\
&= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h} \quad =\cos x\cdot 0-\sin x\cdot 1=-\sin x
\end{aligned}$$

Here are the derivatives of all six of the trigonometric functions.

THEOREM 2.8

1. $\dfrac{d}{dx}[\sin x]=\cos x$ 2. $\dfrac{d}{dx}[\cos x]=-\sin x$

3. $\dfrac{d}{dx}[\tan x]=\sec^2 x$ 4. $\dfrac{d}{dx}[\cot x]=-\ \operatorname{cosec}^2 x$

5. $\dfrac{d}{dx}[\sec x]=\sec x\tan x$ 6. $\dfrac{d}{dx}[\ \operatorname{cosec} x]=-\ \operatorname{cosec} x\cot x$

Example 2.17 1. Let $f(x) = \dfrac{\sin x}{1+\cos x}$. Find $f'(x)$

Solution. $$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{(\cos x)(1+\cos x) + \sin^2 x}{(1+\cos x)^2} \quad \#$$

$\sin^2 x = (\sin x)^2$ / $\sin x^2 \neq (\sin x)^2$

2. Let $f(x) = \sec x$. Find $f''(\frac{\pi}{4})$

Solution. $$f'(x) = \sec x \cdot \tan x$$

$$f''(x) = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)$$

$$= \sec^3 x + \sec x \cdot \tan^2 x$$

$$f''\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) \quad (1)^2$$

$$= \left(\frac{2}{\sqrt{2}}\right)^3 + \frac{2}{\sqrt{2}}$$

$$= \frac{\cancel{8}^{4}}{\cancel{2}\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2} \quad \#$$

$\sec x = \dfrac{1}{\cos x}$

$\cos\left(\frac{\pi}{4}\right) = \cos 45^\circ = \dfrac{\sqrt{2}}{2}$

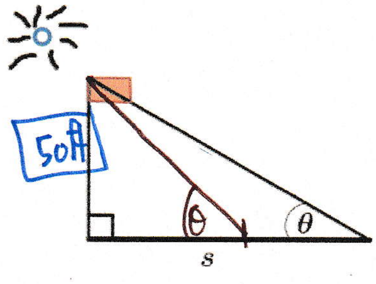
$\sec\left(\frac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}}$

Example 2.18 On a sunny day, a 50 ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun. Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45°$. Express your answer in units of feet/degree.

"หาอัตราการเปลี่ยนแปลงของความยาวของเงาเทียบกับ θ เมื่อ $\theta = 45^\circ$

ให้ความสูงของเสาธงเป็นค่าคงที่ (50 ft)

Solution.



$\therefore \tan\theta = \frac{50}{s} \quad \therefore s = \frac{50}{\tan\theta} = 50\cot\theta$

$\therefore \boxed{s = 50\cot\theta}$

จาก $\left.\frac{ds}{d\theta}\right|_{\theta=45}$ $\quad \frac{ds}{d\theta} = 50(-\operatorname{cosec}^2\theta)$

$\therefore \frac{ds}{d\theta} = -50\operatorname{cosec}^2\theta$

$45° \longrightarrow \frac{\pi}{4}$ radian

$\therefore \left.\boxed{\frac{ds}{d\theta}}\right|_{\theta=\frac{\pi}{4}} = -50\operatorname{cosec}^2\frac{\pi}{4} = -100$ ft/radian

$\operatorname{cosec}\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} = \frac{2}{\sqrt{2}}$

$\operatorname{cosec}^2\frac{\pi}{4} = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2$

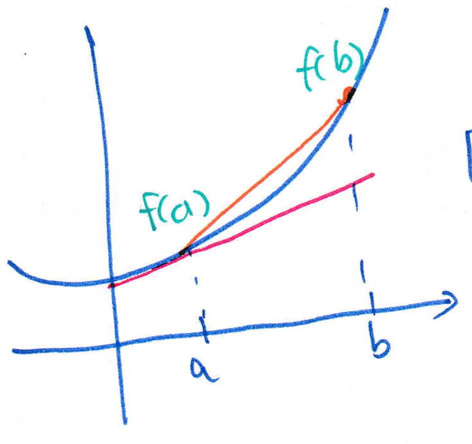
"ความยาวของเงาลดลงด้วยอัตรา 100 ft/radian."

อัตราการเปลี่ยนแปลง

$y = f(x)$

1 อัตราการเปลี่ยนแปลงเฉลี่ย : จาก $x=a$ ถึง $x=b$

$\therefore$ อัตราการเปลี่ยนแปลงเฉลี่ย $= \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$



2 อัตราการเปลี่ยนแปลง ณ ขณะใดขณะหนึ่ง (ณ. $x=a$) : $f'(x)$ เมื่อ $x=a$

Ex $y = \cos x$

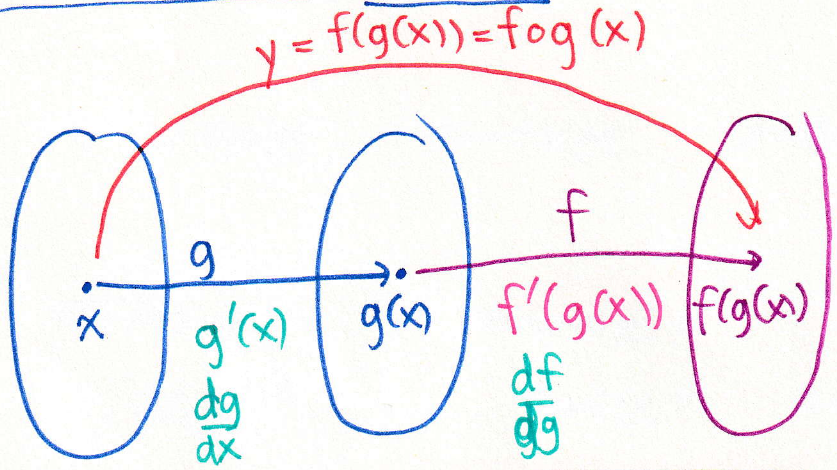
$y = -\sin x$

Ex ให้ $y = 2x+4$

$f(y) = \cos y$

$\cancel{f(x)} =$

$f(x) = \cos(2x+4)$

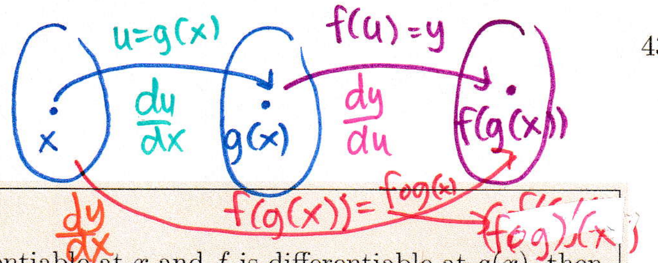


จาก $\boxed{(f\circ g)'(x) = f'(g(x))\, g(x)}$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

กฎลูกโซ่ $\boxed{\text{Chain rule}}$

2.6 The Chain Rule



THEOREM 2.9 (CHAIN RULE) If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

$$\boxed{(f\circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

then $y = f(u)$ and , $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2.19 Find $\frac{dw}{dt}$ if $w = \tan x$ and $x = 4t^3 + t$

Solution. แทน x ใน $w = \tan x \Rightarrow w = \tan(4t^3+t) \Rightarrow$ หา $\boxed{\frac{dw}{dt}}$

$$\underbrace{w \xrightarrow{\frac{dw}{dx}} x \xrightarrow{\frac{dx}{dt}} t}_{\frac{dw}{dt}}$$

$$\therefore \boxed{\frac{dw}{dx} = \sec^2 x}$$

$$\boxed{\frac{dx}{dt} = 12t^2 + 1}$$

$$\boxed{1} \therefore \frac{dw}{dt} = \boxed{\frac{dw}{dx}} \cdot \boxed{\frac{dx}{dt}} = \boxed{\sec^2 x} \cdot \boxed{(12t^2+1)}$$

$$= [\sec^2(4t^3+t)](12t^2+1)$$

$$\boxed{2}\ w = \tan(\underbrace{4t^3+t}_{x}) = \sec^2 x \frac{d}{dt}(x) = \sec^2(4t^3+t) \cdot \frac{d}{dt}(4t^3+t)$$

$$= \sec^2(4t^3+t)[12t^2+1]$$

Example 2.20 Find $\frac{dy}{dt}$ if $y = \sin(4t^3)$

Solution. $y = \sin u$, $u = 4t^3$

$$\underbrace{y \xrightarrow{\frac{dy}{du}} u \xrightarrow{\frac{du}{dt}} t}_{\frac{dy}{dt}}$$

$$\frac{dy}{dt} = \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dt}}$$

$$= \cos u \cdot (12t^2)$$

$$\boxed{\frac{dy}{dt} = [\cos(4t^3)](12t^2)}$$

$$\frac{dy}{dt} = \cos(4t^3) \frac{d}{dt}(4t^3) = \cos(4t^3)[12t^2]$$

Example 2.21 Find $\dfrac{dy}{dx}$ if $y = \sqrt[5]{(x^2 - 9x + 1)^2}$ $\Rightarrow y = (\overbrace{x^2-9x+1}^{u})^{2/5}$

$y = u^{2/5}$; $u = x^2 - 9x + 1$

$$\underbrace{y \xrightarrow{\frac{dy}{du}} u \xrightarrow{\frac{du}{dx}} x}_{\frac{dy}{dx} =}$$

$$\frac{dy}{dx} = \left(\frac{2}{5}u^{-3/5}\right)(2x-9)$$

$$= \frac{2}{5}(x^2-9x+1)^{-3/5}(2x-9)$$

Solution. $y = (\boxed{x^2-9x+1})^{\frac{2}{5}}$

$$\frac{dy}{dx} = \frac{2}{5}(\overbrace{x^2-9x+1}^{u})^{-\frac{3}{5}}\frac{d}{dx}(x^2-9x+1)$$

$$= \frac{2}{5}(x^2-9x+1)^{-\frac{3}{5}}(2x-9)$$

Example 2.22 Find $\dfrac{dw}{dz}$ if $w(z) = \dfrac{1}{\sqrt{9z+1}}$ $\Rightarrow w = (9z+1)^{-\frac{1}{2}}$

Solution.

$$\frac{dw}{dz} = -\frac{1}{2}(9z+1)^{-3/2}\frac{d}{dz}(9z+1)$$

$$= -\frac{9}{2}(9z+1)^{-3/2}$$

Example 2.23 Find $\dfrac{df}{dx}$ if $f(x) = (x^2 - 5)^{-8}$

Solution.