ทบทวน ให้ u, v เป็นแปรที่ขึ้นกับ x.

13 Sep 2019

(1) $\frac{d}{dx}(c) = 0$

(2) $\frac{d}{dx}(ku) = k\frac{du}{dx}$

(3) $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

(4) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(5) $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$

(6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(7) $\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

(8) $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

(9) $\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$

(10) $\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$

(11) $\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u\frac{du}{dx}$

(12) $\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$

(13) $\frac{d}{dx}(\operatorname{cosec} u) = -\operatorname{cosec} u \cot u\frac{du}{dx}$

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(14) $\frac{d}{dx}(a^u) = a^u \ln a\frac{du}{dx} \iff \frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

(15) $\frac{d}{dx}(\log_a u) = \frac{1}{u\ln a}\frac{du}{dx} \iff \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

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(16) $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(17) $\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

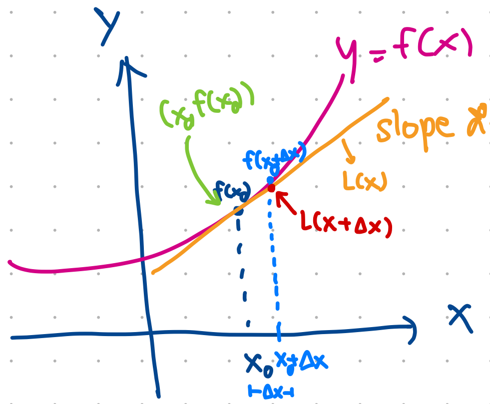
(18) $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}$

(19) $\frac{d}{dx}(\operatorname{arccot} u) = -\frac{1}{1+u^2}\frac{du}{dx}$

(20) $\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

(21) $\frac{d}{dx}(\operatorname{arccosec} u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Local Linear Approximation การประมาณเชิงเส้นเฉพาะที่



ถ้า Δx เล็กๆ เราจะประมาณ $f(x+\Delta x)$ ได้ โดยใช้ $L(x+\Delta x)$

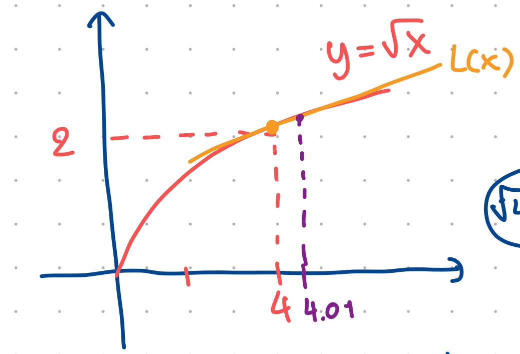
$L(x) - f(x_0) = f'(x_0)(x - x_0)$

$\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$

$x = x_0 + \Delta x$

$L(x_0+\Delta x) = f(x_0) + f'(x_0)(x_0 + \Delta x - x_0)$

Ex ประมาณ $\sqrt{4.01}$



(1) เลือก $f(x) = \sqrt{x}$

(2) เลือก $x_0 = 4$

(3) เลือก $\Delta x = 0.01$

$$\begin{aligned}
\sqrt{4.01} \approx L(4.01) &= f(4) + f'(4)(4 + 0.01 - 4) \\
&= 2 + \tfrac{1}{4}(0.01) \\
&= 2 + (0.25)(0.01) \\
&= 2 + 0.0025 \\
&= 2.0025
\end{aligned}$$

$f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}}$

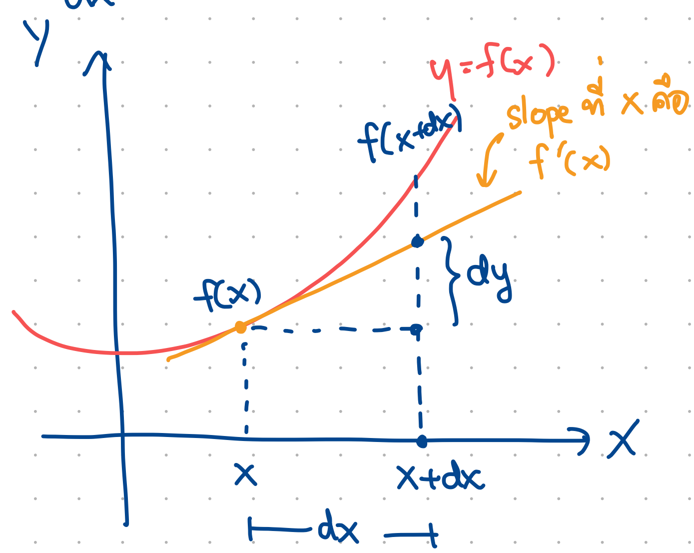
$\therefore f'(4) = \dfrac{1}{4}$

ประมาณ $\sqrt{3.99}$ [เลือก $\Delta x = -0.01$]

ประมาณ $\sqrt{9.01}$

Differential (ดิฟเฟอเรนเชียล) [$y = f(x)$]

$\dfrac{dy}{dx}$ (operator) $\rightarrow f'(x)$



นิยามความหมายของ dx, dy

$\therefore \dfrac{dy}{dx} = f'(x) \Leftarrow dx \neq 0$

$\Downarrow$

$dy = f'(x)\,dx$

Differentials

Let $y = f(x)$ be a differentiable function. The differential of y with respect to x, denoted dy, is $f'(x)dx$, i.e.,

$$dy = f'(x)dx.$$

$f'(x) = \cos(2\pi x)[2\pi]$

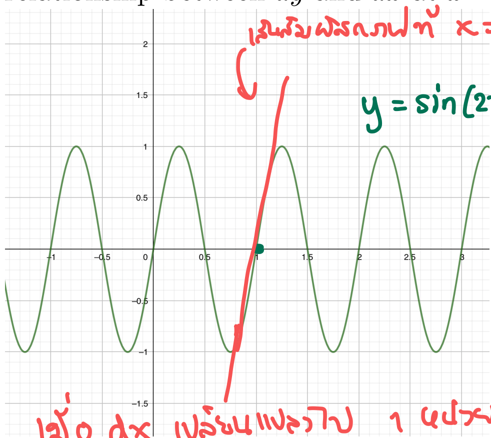
Example 3.23 Find dy when $y = \sin(2\pi x)$ and discuss the relationship between dy and dx at $x = 1$.

$\therefore\ dy = f'(x)dx$

$\therefore\ \boxed{dy = 2\pi\cos(2\pi x)\,dx}$

เมื่อ $x = 1$ $\quad \therefore dy = 2\pi(\cos 2\pi)\,dx$

$\therefore\ dy = 2\pi dx$



เมื่อ dx เปลี่ยนแปลงไป 1 หน่วย
dy เปลี่ยนแปลงไป 2π เท่าของ dx.

Estimation with Differential

Recall that

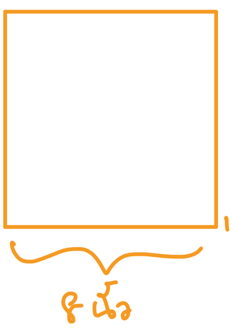
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

So when Δx is small,

$$\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\Delta x \approx f'(x)dx.$$

Example 3.24 Suppose that the side of a square is measured with a ruler to be 8 inches with a measurement error of at most $\pm\frac{1}{64}$ inches. Estimate the error in the computed area of the square.

ทราบว่า ถ้า □ มีความยาวด้าน x แล้ว พ.ท. $y = x^2$



$dy = f'(x)dx$

$\therefore\ dy = (2x)\,dx$

ที่ $x = 8$ นิ้ว $\Rightarrow \boxed{dy = 16dx}$

$-\frac{1}{64} \leq dx \leq \frac{1}{64}$

ถ้าวัดได้ 8.01 นิ้ว พ.ท.เป็น ?

$(8.01)^2 = 64.16$

7.99 นิ้ว พ.ท. = 63.84

เอา 16 คูณทั้งอสมการ:

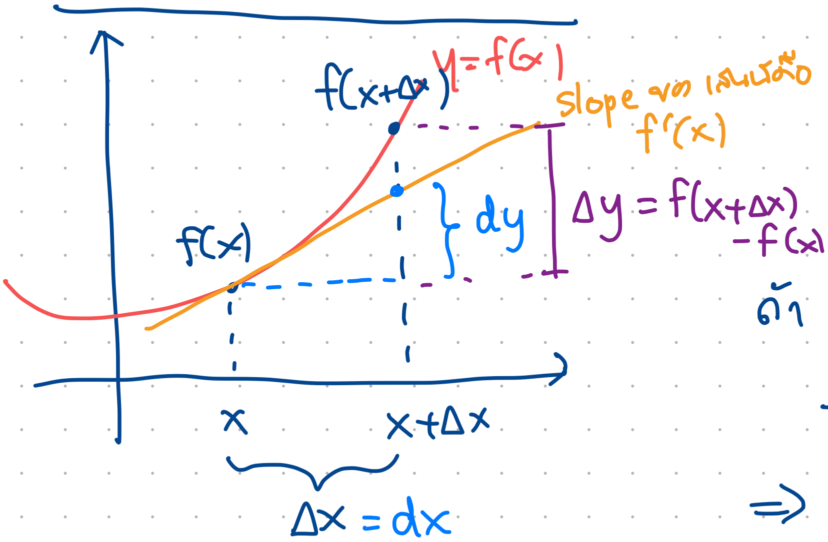
$-\frac{16}{64} \leq 16dx \leq \frac{16}{64}$

$-\frac{1}{4} \leq dy \leq \frac{1}{4}$

(*) เมื่อวัดความยาวด้านคลาดเคลื่อนไป $\pm\frac{1}{64}$ นิ้ว จะได้ว่า
พ.ท. มีความคลาดเคลื่อนไป $\pm\frac{1}{4}$ นิ้ว.

การประมาณค่าโดยใช้ differential

$$\boxed{dy = f'(x)dx}$$



$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

ถ้า Δx น้อยๆ

$$f'(x) \approx \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \Delta y \approx f'(x)\Delta x = f'(x)dx$$

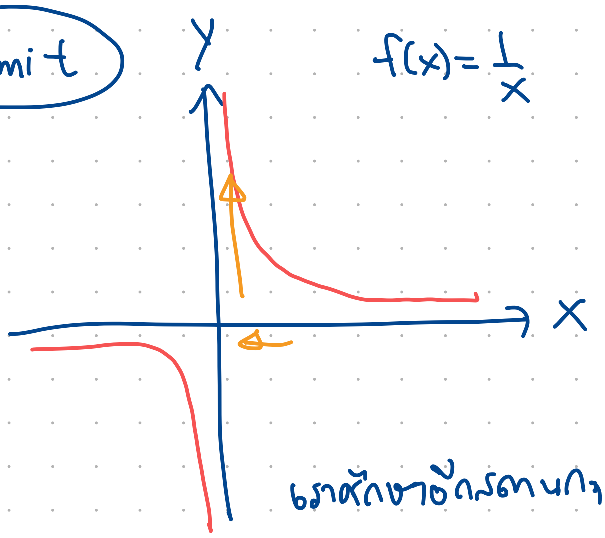
ถ้าเลือก Δx ที่เล็กมากพอ

$$f(x+\Delta x) - f(x) = \Delta y \approx f'(x)dx = dy$$

สรุปว่า $\Delta y = f(x+\Delta x) - f(x) \approx dy = f'(x)\,dx$

Limit

$f(x) = \frac{1}{x}$



$$\lim_{x \to 0^+} f(x) = +\infty$$

เมื่อ x เข้าใกล้ 0 ทางขวามือ ค่าของฟังก์ชันเพิ่มขึ้นอย่างไม่มีขอบเขต

"ลิมิตอนันต์ (Infinity Limit)"

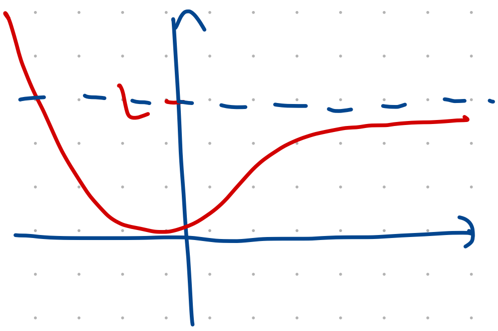
เราศึกษาอีกสถานการณ์ → เมื่อ x เพิ่มขึ้นหรือลดลงอย่างไม่มีขอบเขต แล้ว f(x) จะเป็นอย่างไร ?

$$\lim_{x \to +\infty} f(x) = 0$$
$$\lim_{x \to -\infty} f(x) = 0$$

} **ลิมิตที่อนันต์ (Limit at infinity)**

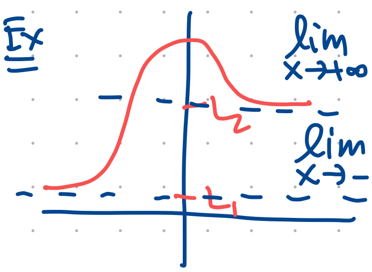
y = a จะเป็นเส้นกำกับแนวนอน (horizontal asymptote) ก็ต่อเมื่อ $\lim_{x \to +\infty} f(x) = a$ หรือ $\lim_{x \to -\infty} f(x) = a$

Ex



$$\lim_{x \to +\infty} f(x) = L$$
$$\lim_{x \to -\infty} f(x) = +\infty$$

Ex



$$\lim_{x \to +\infty} f(x) = L_2$$
$$\lim_{x \to -\infty} f(x) = L_1$$

3.7 Limits at Infinity

In this section, we will discuss the behavior of functions as $x \to -\infty$ and $+\infty$ (the ***end behavior*** of the function).

Example 3.25 Consider the function $f(x) = \frac{1}{x}$.

As $x \to -\infty$, we have $\lim\limits_{x\to-\infty} \frac{1}{x} = \ldots\ldots\ldots\ldots$

	values						conclusion
x	-1	-10	-100	-1000	-10000	$\cdots$	as $x \to -\infty$
$\frac{1}{x}$	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	

As $x \to \infty$, we have $\lim\limits_{x\to\infty} \frac{1}{x} = \ldots\ldots\ldots\ldots$

	values						conclusion
x	1	10	100	1000	10000	$\cdots$	as $x \to \infty$
$\frac{1}{x}$	1	0.1	0.01	0.001	0.0001	$\cdots$	

In general, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound. Then we write

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to -\infty.$$

Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound. Then we write

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to +\infty.$$

Definition 3.5a (Asymptotes) A line $y = L$ is a ***horizontal asymptote*** of thy graph $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to\infty} f(x) = L$$

and a line $x = a$ is a ***vertical asymptote*** of thy graph $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty$$

THEOREM 3.3 Let k, L, M be real numbers and f, g be functions such that $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to+\infty} g(x) = M$. Then,

1. $\lim_{x\to+\infty} k = k$,
2. $\lim_{x\to+\infty} \frac{1}{x^a} = 0$, when a is positive,
3. $\lim_{x\to+\infty} [f(x) \pm g(x)] = L \pm M$,
4. $\lim_{x\to+\infty} f(x)g(x) = LM$,
5. $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, when $M \neq 0$,
6. $\lim_{x\to+\infty} [f(x)]^a = L^a$, when $a > 0$ and $L \geq 0$.

Note that the above theorem also holds in case of x approaching to $-\infty$.

Example 3.26 Find $\lim_{x\to-\infty} \left(\frac{5}{x} - 7\right)$.

$$\lim_{x\to-\infty}\left(\frac{5}{x}-7\right) = 5\underbrace{\lim_{x\to-\infty}\frac{1}{x}}_{0} - \lim_{x\to-\infty} 7$$

$$= -7$$

#

INFINITE LIMITS AT INFINITY

if the values of $f(x)$ decrease or increase without bound as $x \to -\infty$, then we write

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = +\infty,$$

as appropriate. Similarly, if the values of $f(x)$ decrease or increase without bound as $x \to +\infty$, then we write

$$\lim_{x\to+\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to+\infty} f(x) = +\infty,$$

as appropriate.

Example 3.27 Find the following limits.

1. $\lim_{x\to-\infty} x = -\infty$ $\qquad$ $\lim_{x\to+\infty} x = +\infty$

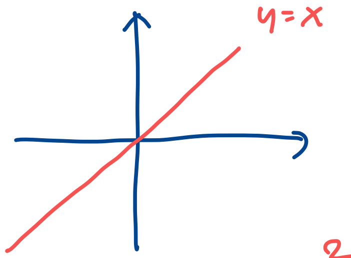
2. $\lim_{x\to-\infty} x^2 = +\infty$ $\qquad$ $\lim_{x\to+\infty} x^2 = +\infty$

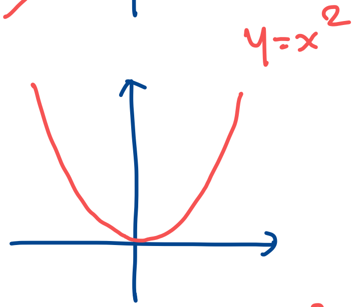
3. $\lim_{x\to-\infty} x^3 = -\infty$ $\qquad$ $\lim_{x\to+\infty} x^3 = +\infty$

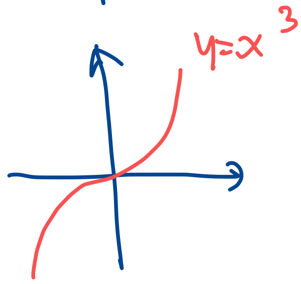
4. $\lim_{x\to-\infty} x^4 = +\infty$ $\qquad$ $\lim_{x\to+\infty} x^4 = +\infty$

5. $\lim_{x\to-\infty} 15x = -\infty$ $\qquad$ $\lim_{x\to+\infty} 15x = +\infty$

6. $\lim_{x\to-\infty} -3x^2 = -\infty$ $\qquad$ $\lim_{x\to+\infty} -3x^2 = -\infty$







⊛ $\lim_{x\to+\infty} x^n = +\infty \;;\; n=1,2,3,\ldots$

$$\lim_{x\to-\infty} x^n = \begin{cases} -\infty & ; n=1,3,5,\ldots \\ +\infty & ; n=2,4,6,\ldots \end{cases}$$

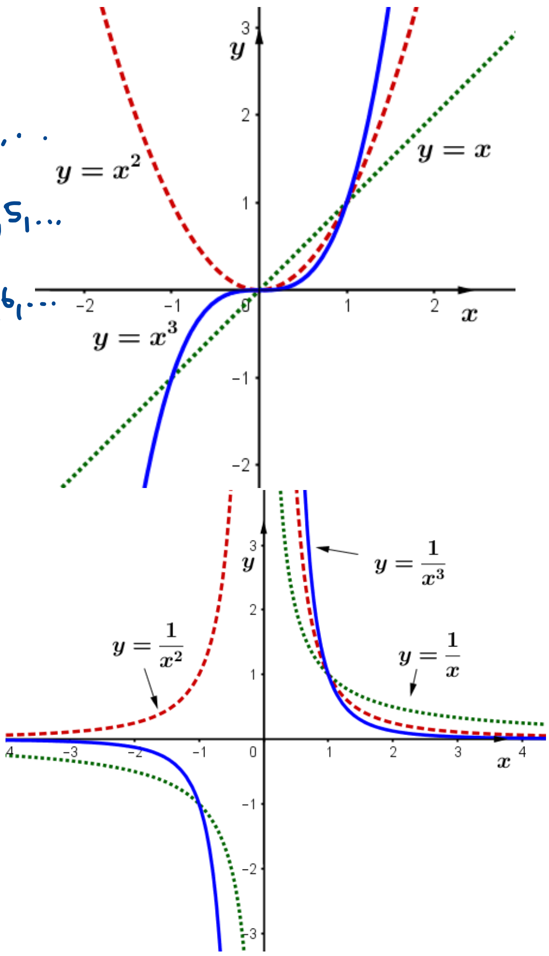


Figure 3.3: Graphs of polynomails x, x^2, x^3 and rational functions $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$

The end behavior of polynomials

The end behavior of a polynomial matches the end behavior of its highest degree term.

Example 3.28 Find the following limits.

1. $\lim_{x\to-\infty}(5x^4 - 10x^3 + 897x + 84000) = \lim_{x\to-\infty} 5x^4 = +\infty$

   $+\infty \quad +\infty \quad -\infty$ ← indeterminate form

2. $\lim_{x\to+\infty}(-x^3 + 650x^2 + 7x - 1) = \lim_{x\to+\infty} -x^3 = -\infty$

Example 3.29 Find the following limits.

1. $\lim_{x\to-\infty}\dfrac{24x-9}{8x+11}$.

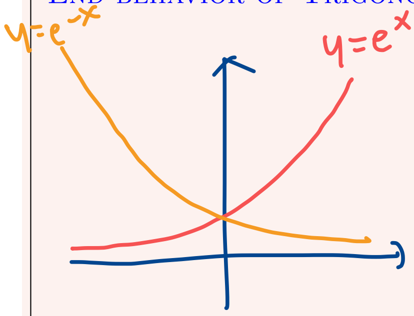
2. $\lim_{x\to+\infty}\dfrac{3x^2-5x}{8x^6+14x-1}$.

Limits involving radicals

Example 3.30 Find the following limits.

1. $\lim_{x\to-\infty}\dfrac{\sqrt{4x^2+3x+2}}{2x-7}$.

End behavior of Trigonometric, Exponential and Logarithmic Functions



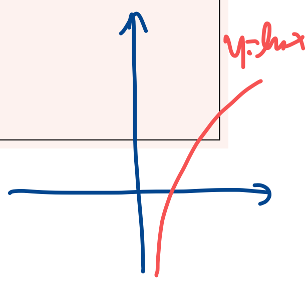
$$\left.\begin{array}{l}\lim_{x\to-\infty} \sin x,\\ \lim_{x\to+\infty} \sin x,\\ \lim_{x\to-\infty} \cos x,\\ \lim_{x\to+\infty} \cos x\end{array}\right\}$$ ไม่มี ....

(ฟังก์ชันแกว่ง (oscillating) ระหว่าง −1, 1)

$\lim_{x\to-\infty} e^{x} =$ 0 $\qquad$ $\lim_{x\to+\infty} e^{x} =$ $+\infty$

$\lim_{x\to-\infty} e^{-x} =$ $+\infty$ $\qquad$ $\lim_{x\to+\infty} e^{-x} =$ 0

$\lim_{x\to 0^+} \ln x =$ $-\infty$ $\qquad$ $\lim_{x\to+\infty} \ln x =$ $+\infty$



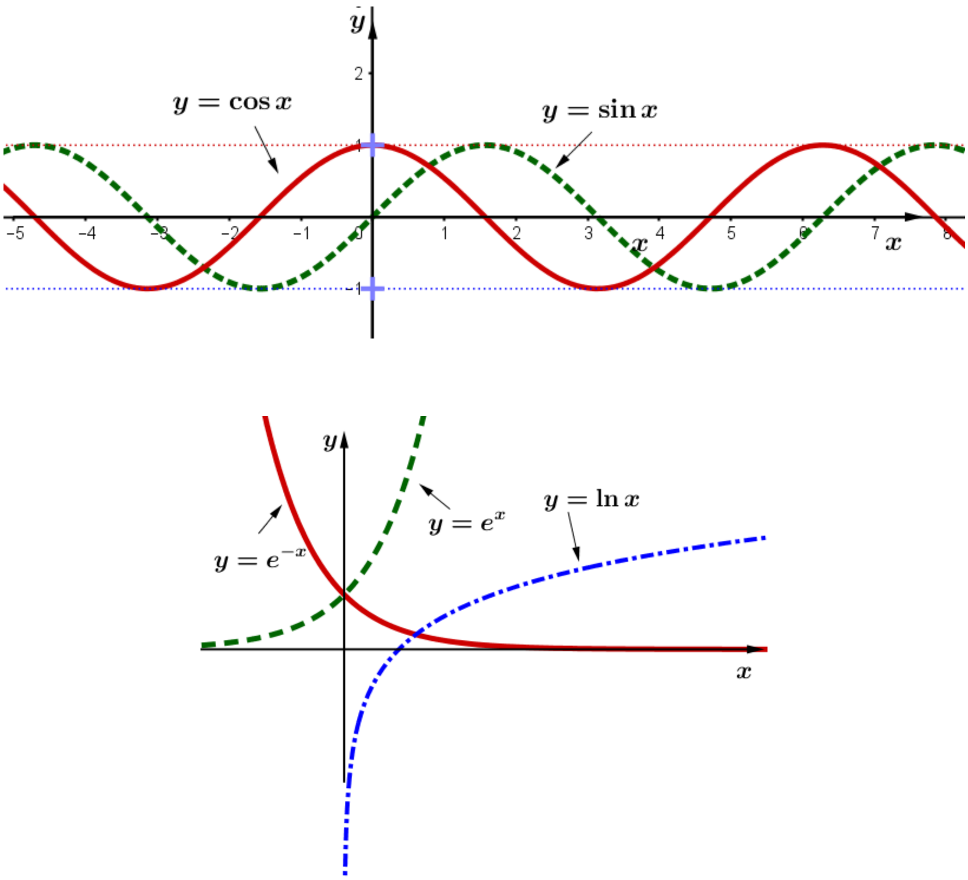


Figure 3.4: Graphs of $\sin x \cos x, e^{x}, e^{-x}$ and $\ln x$