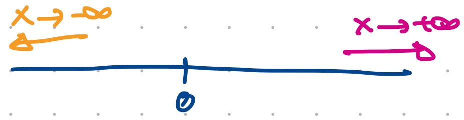
ทบทวน ลิมิตที่อนันต์

$\lim_{x\to+\infty} f(x)$, $\lim_{x\to-\infty} f(x)$



$$\lim_{x\to+\infty} e^x = +\infty$$

$$\boxed{\lim_{x\to-\infty} e^x = 0}$$

$y=0$ เป็น horizontal asymptote ของ $y=e^x$

$$\lim_{x\to+\infty} e^{-x} = 0 \Rightarrow \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} e^{-x} = +\infty \Rightarrow \lim_{x\to-\infty} \frac{1}{e^x} = \lim_{x\to-\infty} e^{-x} = +\infty$$

Ex $\lim_{x\to-\infty} (5x^4-10x^3+897x+8400) = \lim_{x\to-\infty} x^4\left(5-\cancel{\frac{10}{x}}^{0}+\cancel{\frac{897}{x^3}}^{0}+\cancel{\frac{8400}{x^4}}^{0}\right) = +\infty$

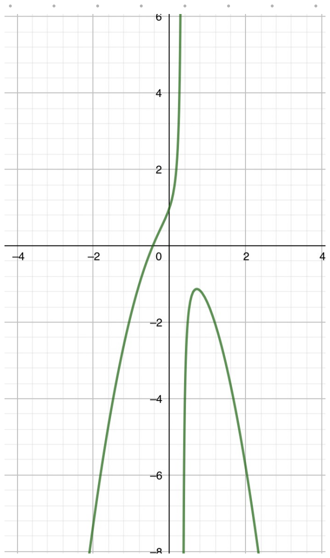
$\lim_{x\to+\infty} (-x^3+650x^2+7x-1) = \lim_{x\to+\infty} x^3\left(-1+\cancel{\frac{650}{x}}^{0}+\cancel{\frac{7}{x^2}}^{0}-\cancel{\frac{1}{x^3}}^{0}\right) = -\infty$

Ex $\lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim_{x\to+\infty} \frac{\frac{5x^3}{x^3}-\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{1}{x^3}-\frac{3x}{x^3}} = \lim_{x\to+\infty} \frac{5-\frac{2}{x}+\frac{1}{x^3}}{\frac{1}{x^3}-\frac{3}{x^2}}$

พิจารณาใหม่ $\lim_{x\to+\infty} \frac{5x^2-2x+\frac{1}{x}}{\frac{1}{x}-3}$

① $\lim_{x\to+\infty} 5x^2-2x = +\infty$

② $\lim_{x\to+\infty} \frac{1}{x} = 0$



$$\therefore \lim_{x\to+\infty} \frac{\overbrace{5x^2-2x}^{+\infty}+\cancel{\frac{1}{x}}^{0}}{\cancel{\frac{1}{x}}^{0}-3} = -\infty$$

The end behavior of polynomials

The end behavior of a polynomial matches the end behavior of its highest degree term.

Example 3.28 Find the following limits.

1. $\lim_{x\to-\infty}(5x^4 - 10x^3 + 897x + 84000) =$

2. $\lim_{x\to+\infty}(-x^3 + 650x^2 + 7x - 1) =$

ดูว่าดีกรีมากสุดในฟังก์ชันตรรกยะคือใคร ให้เอาดีกรีสูงสุด
หารทุกพจน์ แล้วจัดรูป ดูพฤติกรรม

Example 3.29 Find the following limits.

1. $\lim_{x\to-\infty}\frac{24x-9}{8x+11}$. $= \lim_{x\to-\infty}\frac{\cancel{x}(24-\frac{9}{x})}{\cancel{x}(8+\frac{11}{x})} = \lim_{x\to-\infty}\frac{24-\frac{9}{x}\to 0}{8+\frac{11}{x}\to 0} = \frac{24}{8} = 3$

2. $\lim_{x\to+\infty}\frac{3x^2-5x}{8x^6+14x-1}$. $= \lim_{x\to+\infty}\frac{\frac{3x^2}{x^6}-\frac{5x}{x^6}}{\frac{8x^6}{x^6}+\frac{14x}{x^6}-\frac{1}{x^6}} = \lim_{x\to+\infty}\frac{\frac{3}{x^4}\to 0-\frac{5}{x^5}\to 0}{8+\frac{11}{x^5}\to 0-\frac{1}{x^6}\to 0} = 0$

Limits involving radicals

Example 3.30 Find the following limits.

1. $\lim_{x\to-\infty}\frac{\sqrt{4x^2+3x+2}}{2x-7}$. $= \lim_{x\to-\infty}\frac{\sqrt{x^2(4+\frac{3}{x}+\frac{2}{x^2})}}{x(2-\frac{7}{x})}$

$= \lim_{x\to-\infty}\frac{|x|\sqrt{(4+\frac{3}{x}+\frac{2}{x^2})}}{x(2-\frac{7}{x})}$

$= \lim_{x\to-\infty}\frac{(-\cancel{x})\sqrt{4+\frac{3}{x}\to 0+\frac{2}{x^2}\to 0}}{\cancel{x}(2-\frac{7}{x}\to 0)}$

$= -\frac{\sqrt{4}}{2} = -1$ #

$\boxed{\sqrt{x^2} = |x|}$

$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$

Ex จงหา $\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}$

$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to+\infty}\frac{\sqrt{x^2(1+\frac{2}{x^2})}}{x(3-\frac{6}{x})}$

$= \lim_{x\to+\infty}\frac{|x|\sqrt{(1+\frac{2}{x^2})}}{x(3-\frac{6}{x})}$

$= \lim_{x\to+\infty}\frac{\cancel{x}\sqrt{1+\frac{2}{x^2}\to 0}}{\cancel{x}(3-\frac{6}{x}\to 0)} = \frac{1}{3}$

2. $\lim_{x\to+\infty}\left(x^2-\sqrt{x^4+9}\right)$. $= \lim_{x\to+\infty} \left(x^2-\sqrt{x^4+9}\right)\dfrac{\left(x^2+\sqrt{x^4+9}\right)}{x^2+\sqrt{x^4+9}}$

$= \lim_{x\to+\infty} \dfrac{(x^2)^2-\left(\sqrt{x^4+9}\right)^2}{x^2+\sqrt{x^4+9}}$

$= \lim_{x\to+\infty} \dfrac{\cancel{x^4}-(\cancel{x^4}+9)}{x^2+\sqrt{x^4+9}}$

$= \lim_{x\to+\infty} -\dfrac{9}{x^2+\sqrt{x^4+9}}$

$= 0$

Conjugate ของ $x^2-\sqrt{x^4+9}$ คือ $x^2+\sqrt{x^4+9}$

$(+\infty)+\infty = +\infty$

$(\infty)\cdot(\infty) = +\infty$

$+\infty - \infty$ ระบุไม่ได้

$\Downarrow$

รูป indeterminate

3. $\lim_{x\to+\infty}\dfrac{1}{\sqrt{x^4+3x^2}-x^2}$. $= \lim_{x\to+\infty} \dfrac{1}{\sqrt{x^4+3x^2}-x^2}\cdot\dfrac{\left(\sqrt{x^4+3x^2}+x^2\right)}{\left(\sqrt{x^4+3x^2}+x^2\right)}$

$= \lim_{x\to+\infty} \dfrac{\sqrt{x^4+3x^2}+x^2}{(\cancel{x^4}+3x^2)-\cancel{x^4}}$

$= \lim_{x\to+\infty} \dfrac{\overset{\infty}{\sqrt{x^4+3x^2}}+\overset{\infty}{x^2}}{\underset{\infty}{3x^2}}$

$\boxed{\dfrac{\infty}{\infty}}$ — indeterminate form.

$= \lim_{x\to+\infty} \dfrac{\sqrt{x^4\left(1+\frac{3}{x^2}\right)}+x^2}{3x^2}$

$= \lim_{x\to+\infty} \dfrac{|x^2|\sqrt{1+\frac{3}{x^2}}+x^2}{3x^2}$

$|x^2| = \begin{cases} x^2 ; & x^2 \geq 0 \\ \cancel{-x^2 ;} & \cancel{x^2 < 0} \end{cases}$

$= \lim_{x\to+\infty} \dfrac{x^2\sqrt{1+\frac{3}{x^2}}+x^2}{3x^2}$

$= \lim_{x\to+\infty} \dfrac{\cancel{x^2}\left(\sqrt{1+\frac{3}{x^2}}+1\right)}{3\cancel{x^2}}$

$= \lim_{x\to+\infty} \dfrac{\sqrt{1+\cancel{\frac{3}{x^2}}}+1}{3} = \dfrac{2}{3}$ #

Ex $\displaystyle\lim_{x\to+\infty} \frac{53x+1}{32x+4}$ $\quad \frac{\infty}{\infty}$

$\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$ $\quad \frac{0}{0}$

$\displaystyle\lim_{x\to 0} \frac{3x-\sin x}{x}$

Indeterminate form รูปแบบไม่กำหนด

$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\cdot\infty, 1^{\infty}, \infty^{0}, 0^{0}$

1. จัดรูปให้ปลดเปลื้องจาก indeterminate form
   - conjuate
   - แยกตัวประกอบ
2. กฎโลปิตาล L'Hôpital Rule.

Thm 3.4 ให้ f, g เป็นฟังก์ชันที่หาอนุพันธ์ได้บนช่วงเปิดที่บรรจุ $x=a$ โดยที่ $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

ถ้า $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ มีค่า แล้ว

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

เป็นจริง สำหรับ $x\to a^+$, $x\to a^-$

Ex $\displaystyle\lim_{x\to 0} \frac{x+b}{x+2} \overset{\text{L'Hôpital}}{\cancel{=}} \cancel{\lim_{x\to 0} \frac{1}{1} = 1}$ #

⊛ $\lim_{x\to 0} x+b = 1 \neq 0$, $\lim_{x\to 0} x+2 = 1 \neq 0$

ใช้ L'Hôpital ไม่ได้

Ex $\displaystyle\lim_{x\to 0^+} \frac{2+\overset{0}{x}}{1-\cos x} \cancel{\overset{\text{L'Hôpital}}{=} \lim_{x\to 0^+} \frac{1}{\sin x} = +\infty}$

ผิด เพราะ $\lim_{x\to 0^+} 2+x \neq 0$

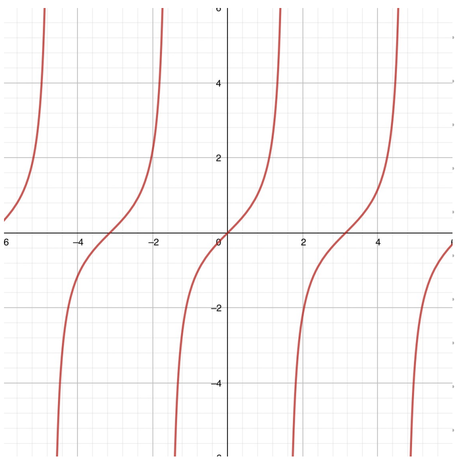
เมื่อ $x\to 0^+$ แล้ว $\cos x \to 1^-$

$\therefore 1-\cos x \to 0^+$

$\therefore$ นั่นคือ $\displaystyle\lim_{x\to 0^+} \frac{2+x}{1-\cos x} = +\infty$

Ex $\displaystyle\lim_{x\to 0^+} \frac{\tan x}{x^2} \; \frac{0}{0} = \lim_{x\to 0^+} \frac{\sec^2 x}{2x} \; \frac{1}{0^+} = +\infty$ เมื่อ $x\to 0^+$ ได้ $2x \to 0^+$

$\therefore \frac{\sec^2 x}{2x} \to +\infty$



3.8 Indeterminate Forms

3.8.1 L'Hôpital's Rule

"เป็นจริงสำหรับ $x \to a^+$ / $x \to a^-$ / $x \to +\infty$ / $x \to -\infty$"

$\frac{0}{0}$

Theorem 3.4 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the theorem also works for one-sided limits.

Example 3.31 Find $\lim_{x\to 0} \frac{3x - \sin x}{x}$. $\frac{0}{0}$

Check $\lim_{x\to 0} 3x - \sin x = 0$

$\lim_{x\to 0} x = 0$

$$\therefore \lim_{x\to 0} \frac{3x-\sin x}{x} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\frac{d}{dx}(3x - \sin x)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{3-\cos x}{1} = 3 - 1 = 2$$

Ex $\lim_{x\to 2} \frac{x^2-4}{x-2}$ $\left(= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4 \right)$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{2x}{1} = 4$$

Theorem 3.5 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the theorem also works for one-sided limits.

Example 3.32 Find $\lim_{x\to\infty} \frac{2x+3}{5x+7}$. $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to+\infty} \frac{\frac{d}{dx}(2x+3)}{\frac{d}{dx}(5x+7)} = \lim_{x\to+\infty} \frac{2}{5} = \frac{2}{5}$$

Other Indeterminate Forms

Solving other indeterminate forms require us to change the limit so that we can use L'hospital's rule.

Example 3.33 Find $\lim\limits_{x\to\infty} x\tan\dfrac{1}{x}$

Example 3.34 Find $\lim\limits_{x\to 0^+}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

In case the limit of the form $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$, apply logarithm to the limit.

Example 3.35 Find $\lim\limits_{x\to 0^+}(\sin x)^x$.

Failures of L'Hopital's Rule

Sometimes using L'hospital's Rule will not solve the problem as in following example.

Example 3.36 Find $\lim\limits_{x\to(\pi/2)^-} \dfrac{\sec x}{\tan x}$.

Failed Solution First, let find out what happens if we use L'Hopital's rule.

Notice that $\lim\limits_{x\to(\pi/2)^-} \sec x = \infty = \lim\limits_{x\to(\pi/2)^-} \tan x$. Therefore, it is possible to use L'Hoslpital's rule.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{\frac{d}{dx}\sec x}{\frac{d}{dx}\tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}.$$

Using L'Hopital's rule again, we will have

$$\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}.$$

So we return to the problem at the beginning and nothing is solved!

Good Solution ***Try simplifying the expression first.***

$$\frac{\sec x}{\tan x} = \frac{1/\cos x}{\sin x/\cos x} = \frac{1}{\sin x}.$$

Therefore,

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{1}{\sin x} = 1.$$