$\boxed{\int u\,dv}$

diff ต่อๆ	integrate ต่อๆ
u	dv
du	$v = \int \text{....}\, dv$

$$\int u\,dv = uv - \int v\,du$$

expo trig

LIATE

Example 6.4 Evaluate $\int e^x \cos x \, dx$.

เลือก $u = \cos x$; $dv = e^x dx$

$du = -\sin x \, dx$; $v = \int e^x dx = e^x$

$$\therefore \int e^x \cos x \, dx = e^x \cos x - \int e^x(-\sin x)\,dx$$

$$= e^x \cos x + \int e^x \sin x \, dx \quad \text{———} (*)$$

พิจารณา $\int e^x \sin x \, dx$ เลือก $u_1 = \sin x$; $dv_1 = e^x dx$

$du_1 = \cos x \, dx$; $v_1 = e^x$

$$\therefore \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \quad \text{———} (**)$$

แทน (**) ใน (*) $\therefore \int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\therefore \int e^x \cos x \, dx = \frac{1}{2}\left[e^x \cos x + e^x \sin x\right] + C \quad \#$$

[อินทิเกรต by parts 2 ครั้ง แล้วจะวนซ้ำ]

Example 6.5 Evaluate $\int \arctan x \, dx$.

$u = \arctan x$; $dv = dx$

$du = \frac{1}{1+x^2} dx$; $v = x$

$$\therefore \int \arctan x \, dx = x \arctan x - \boxed{\int \frac{x}{1+x^2} dx}$$

$u = 1 + x^2$

$\therefore du = 2x \, dx$

$dx = \frac{du}{2x}$

พิจารณา $\int \frac{x}{1+x^2} dx = \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int \frac{1}{u} du$

$= \frac{\ln|u|}{2}$

$= \frac{1}{2}\ln|1+x^2|$

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2}\ln(x^2+1) + C \quad \#$$

การอินทิเกรตฟังก์ชันตรีโกณมิติบางแบบ

6.3 Integrating Trigonometric Functions

$\sin^2 x + \cos^2 x = 1$ — ⊛

We start the section by reviewing important trigonometric identities as following:

เอา $\sin^2 x$ หารตลอด

⊛ $1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$\Rightarrow 1 + \cot^2 x = \operatorname{cosec}^2 x$

เอา $\cos^2 x$ หารตลอด

$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x = \sec^2 x - 1$
$\sin 2x = 2 \sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

6.3.1 Integrating Products of Sines and Cosines

If m and n are positive integers, the integral $\int \sin^m x \cos^n x \, dx$ can be evaluated by one of the following procedures, depending on whether m and n are even or odd.

$\int \sin^m x \cos^n x \, dx$	Procedure	relevant identity
m odd	set $\sin^m x = \sin^{m-1} x \sin x$	$\sin^2 x = 1 - \cos^2 x$
n odd	set $\cos^n x = \cos^{n-1} x \cos x$	$\cos^2 x = 1 - \sin^2 x$
m and n even	set $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or set $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Trick
① เราแยกตัวเป็นเลขคี่ออกเป็น กำลังหนึ่ง และ กำลังเลขคู่
② เราใช้เอกลักษณ์มาช่วยจัด $\sin^2 x = 1 - \cos^2 x$

Example 6.6 Evaluate $\int \sin^3 x \, dx$.

$$\int \sin^3 x \, dx = \int \sin x \cdot \sin^2 x \, dx$$
$$= \int \sin x (1 - \cos^2 x) dx$$

ให้ $u = \cos x$; $du = -\sin x \, dx \Rightarrow dx = \frac{du}{-\sin x}$

$$= \int \cancel{\sin x} (1 - u^2) \frac{du}{-\cancel{\sin x}}$$
$$= \int (u^2 - 1) du$$
$$= \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C \quad \#$$

Example 6.7 Evaluate $\int \cos^5 x \, dx$.

$$\int \cos^5 x \, dx = \int \cos x \cdot \cos^4 x \, dx$$
$$= \int \cos x (\cos^2 x)^2 dx$$
$$= \int \cos x (1 - \overbrace{\sin^2 x}^{u^2})^2 dx \quad (du)$$

ให้ $u = \sin x$; $du = \cos x \, dx$

$$= \int (1 - u^2)^2 du$$
$$= \int (1 - 2u^2 + u^4) du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \quad \#$$

$\sin^2 x + \cos^2 x = 1$

Example 6.8 Evaluate $\int \sin^2 x \cos^3 x \, dx$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \underbrace{\cos x \, dx}_{du}$$

ให้ $u = \sin x$; $du = \cos x \, dx$

$$= \int u^2 (1 - u^2) \, du$$

$$= \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \quad \#$$

ถ้าเปลี่ยน $\sin^2 x$ เป็น $\cos^2 x$ อาจจะยากขึ้น เพราะต้องแยกอีกรอบ

$$\boxed{\begin{aligned} &= \int (1 - \cos^2 x) \cos^2 x \cdot \cos x \, dx \\ &= \int (\cos^3 x - \cos^5 x) \, dx \end{aligned}}$$

Example 6.9 Evaluate $\int \cos^{1/3} x \sin^3 x \, dx$.

$$\int \cos^{1/3} x \sin^3 x \, dx = \int \cos^{1/3} x \sin^2 x \cdot \sin x \, dx$$

$$= -\int \cos^{1/3} x (1 - \cos^2 x) \underbrace{(-\sin x \, dx)}_{du}$$

ให้ $u = \cos x$; $du = -\sin x \, dx$

$$= -\int u^{1/3} (1 - u^2) \, du$$

$$= \int u^{1/3} (u^2 - 1) \, du$$

$$= \int (u^{7/3} - u^{1/3}) \, du$$

$$= \frac{u^{10/3}}{\frac{10}{3}} - \frac{u^{4/3}}{\frac{4}{3}} + C = \frac{3}{10} u^{10/3} - \frac{3}{4} u^{4/3} + C$$

$$= \frac{3}{10} \cos^{10/3} x - \frac{3}{4} \cos^{4/3} x + C \quad \#$$

Example 6.10 Evaluate $\int (1 + \sin x)^2 \, dx$.

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x = \sec^2 x - 1$
$\sin 2x = 2 \sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\int (1 + \sin x)^2 \, dx = \int (1 + 2 \sin x + \sin^2 x) \, dx$$

$$= x - 2 \cos x + \int \sin^2 x \, dx$$

$$= x - 2 \cos x + \int \frac{1}{2} (1 - \cos 2x) \, dx$$

(ใช้เอกลักษณ์ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$)

$$= x - 2 \cos x + \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= x - 2 \cos x + \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + C \quad \#$$

$\int \cos 2x \, dx$

$u = 2x$; $du = 2 \, dx$

$dx = \frac{du}{2}$

$$\therefore \int \cos 2x \, dx = \int \frac{\cos u}{2} \, du$$

$$= \frac{\sin 2x}{2} + C$$

จำเพิ่ม

$$\boxed{\begin{aligned} \int \cos mx \, dx &= \frac{\sin mx}{m} + C \\ \int \sin mx \, dx &= -\frac{\cos mx}{m} + C \end{aligned}}$$

6.3.2 Integrating Products of Sines and Cosines with Different Angles

Integrals of the form

$$\int \sin mx \cos nx \; dx, \int \sin mx \sin nx \; dx, \int \cos mx \cos nx \; dx$$

$\int \sin mx \, dx = -\frac{\cos mx}{m} + C$

$\int \cos mx \, dx = \frac{\sin mx}{m} + C$

can be evaluated using the following identities:

$\sin(mx)\cos(nx) = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$ — 1
$\sin(mx)\sin(nx) = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$ — 2
$\cos(mx)\cos(nx) = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}$ — 3

Example 6.11 Evaluate $\int \sin 3x \;\cos 5x \; dx$.

$\boxed{\begin{array}{l}\sin(-x) = -\sin x \\ \cos(-x) = \cos x\end{array}}$ (*)

$$\int \sin 3x \cos 5x \, dx = \int \frac{1}{2}[\sin(3x+5x) + \sin(3x-5x)]\,dx \quad (1)$$

$$= \frac{1}{2}\int[\sin 8x + \sin(-2x)]\,dx$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8} - \frac{\cos(-2x)}{(-2)}\right) + C$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos(-2x)}{2}\right) + C$$

$$= -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$

Example 6.12 Evaluate $\int \sin 3x \;\sin \frac{5x}{2} \; dx$.

$$\int \sin 3x \sin\frac{5x}{2}\,dx = \int \frac{1}{2}\left[\cos\left(3x - \frac{5x}{2}\right) - \cos\left(3x + \frac{5x}{2}\right)\right]dx \quad (2)$$

$$= \frac{1}{2}\int\left[\cos\left(\frac{x}{2}\right) - \cos\left(\frac{11x}{2}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{1}{2}} - \frac{\sin\left(\frac{11x}{2}\right)}{\frac{11}{2}}\right] + C$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\sin\left(\frac{x}{2}\right) - \frac{\cancel{2}}{11}\sin\left(\frac{11x}{2}\right)\right] + C$$

$$= \sin\left(\frac{x}{2}\right) - \frac{1}{11}\sin\left(\frac{11x}{2}\right) + C \quad \#$$