ทบทวน

① u-substitution

② by parts

$\int u\,dv \longrightarrow \begin{matrix} u, & dv \\ du, & v \end{matrix}$

$\Downarrow$

$\int u\,dv = uv - \int v\,du$

$\int x\arctan x\,dx \qquad u = \arctan x,\quad dv = x\,dx$

$\int \ln(x^2+4)\,dx \qquad u = \ln(x^2+4),\quad dv = dx$

③ Some trigonometric functions

- $\int \sin^m x \cos^n x\,dx$
- $\int \sin mx \cos nx\,dx$

④ Trigonometric Substitution

$\pm\sqrt{a^2-u^2} \Leftrightarrow u = a\sin\theta$

$\pm\sqrt{a^2+u^2} \Leftrightarrow u = a\tan\theta$

$\pm\sqrt{u^2-a^2} \Leftrightarrow u = a\sec\theta$

⑤ การอินทิเกรตแบบแยกเศษส่วนย่อย

Integration by partial fractions

$\boxed{f(x)} = \dfrac{p(x)}{q(x)}$; $p(x), q(x)$ เป็นพหุนาม

ฟังก์ชันตรรกยะ (rational function)

$q(x)$ จะแยกตัวประกอบได้เป็นพหุนามดีกรี 1 / 2 เสมอ.

idea

$$\int \frac{2}{(x+1)(x-2)}\,dx = \int \left(\frac{A}{x+1} + \frac{B}{x-2}\right) dx$$

การจะแยกเป็นเศษส่วนย่อยได้ เราจะต้องตรวจสอบเสมอว่า $\deg(p(x)) < \deg(q(x))$

ถ้า $f(x) = \dfrac{p(x)}{q(x)}$ ซึ่ง $\deg(p(x)) < \deg(q(x))$

เราจะเรียก $f(x)$ ว่า ฟังก์ชันตรรกยะแท้ (proper rational f^{n})

ถ้า $f(x)$ ไม่เป็นฟังก์ชันตรรกยะแท้ เราจะทำมันให้เป็นฟังก์ชันตรรกยะแท้

Ex $f(x) = \dfrac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}$; $p(x)$, $q(x)$

$\deg(p(x)) = 4$

$\deg(q(x)) = 2$

ตั้งหารยาว

$$\begin{array}{r|l} & 3x^2 \qquad\quad +1 \quad \leftarrow \text{ผลหาร} \\ x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \quad \leftarrow \text{ตัวตั้ง} \\ & \pm3x^4 \pm 3x^3 \mp 6x^2 \\ \hline & \qquad\qquad x^2+x-1 \\ & \qquad\qquad \pm x^2 \pm x \mp 2 \\ \hline & \qquad\qquad\qquad 1 \quad \leftarrow \text{เศษ} \end{array}$$

↑ ตัวหาร

$$f(x) = \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

Case 1 $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$, $P(x) = \ldots$ $\quad (a_i \neq a_j,\ i \neq j)$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

$\downarrow$

Case 2 $Q(x) = (x-a_1)^2(x-a_2)^3(x-a_3)$

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a_1)} + \frac{B}{(x-a_1)^2} + \frac{C}{(x-a_2)} + \frac{D}{(x-a_2)^2} + \frac{E}{(x-a_2)^3} + \frac{F}{x-a_3}$$

Case 3 $Q(x) = (ax^2+bx+c)^2(x-a_1)^2(x-a_2)$ เงื่อนไข : ax^2+bx+c แยกตัวประกอบไม่ได้แล้ว

$\Downarrow$

$b^2-4ac<0$

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{E}{x-a_1} + \frac{F}{(x-a_1)^2} + \frac{G}{x-a_2}$$

"แยกตัวประกอบให้สุด!!"

$\boxed{\text{ll}}$ $$\frac{3x-1}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$$

$\boxed{1}$ $$\frac{1}{x^2-3x-4} = \frac{1}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

$\boxed{2}$ $$\frac{2x-3}{x^3-x^2} = \frac{2x-3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$\boxed{1}$ $$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\boxed{3}$ $$\frac{2x^2-1}{(4x-1)(x^2+1)} = \frac{A}{4x-1} + \frac{Bx+C}{x^2+1}$$

6.5 Integrating Rational Functions by Partial Fractions

Recall that a rational function is a function that can be written as a quotient of two polynomials. Assume that $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where $P(x)$ and $Q(x)$ are polynomials. If $\deg P(x) < \deg Q(x)$, then $f(x)$ is said to be ***proper***. If $\deg P(x) \geq \deg Q(x)$, then $f(x)$ is said to be ***improper*** .

We now find the form of partial fraction decomposition of a proper rational function $f(x) = \dfrac{P(x)}{Q(x)}$. Elementary algebra tells us that $Q(x)$ has only two types of irreducible factors which are of degree 1 or degree 2. Therefore, the partial fraction decomposition of $f(x)$ can be determined by using the following rules, **Linear factor and Quadratic factor rules**.

> **Linear factor rule**: For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:
> $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ \ldots + \dfrac{A_m}{(ax+b)^m}$, where $A_i \quad (i = 1, 2, \ldots, m)$ are constants.

Example 6.16 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$. $= \displaystyle\int \frac{1\,dx}{(x+2)(x-1)}$ $\quad \deg(P(x)) = 0$, $\deg(Q(x)) = 2$

สมมติว่า $\dfrac{1}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{(x-1)}$ $\Rightarrow$ หาว่า A, B คือใคร

คูณ $(x+2)(x-1)$; $\quad 1 = A(x-1) + B(x+2)$

$1 = Ax - A + Bx + 2B$

$0x + 1 = (A+B)x + (-A+2B)$

เทียบสัมประสิทธิ์

$$\left.\begin{aligned} A + B &= 0 \\ -A + 2B &= 1 \end{aligned}\right\}$$

$\therefore A = -B$

$\Rightarrow -(-B) + 2B = 1$

$3B = 1 \Rightarrow B = \frac{1}{3}$

$\therefore A = -\frac{1}{3}$

$$\therefore \frac{1}{(x+2)(x-1)} = -\frac{1}{3}\cdot\frac{1}{x+2} + \frac{1}{3}\cdot\frac{1}{x-1}$$

อินทิเกรต

$$\int \frac{1}{(x+2)(x-1)}dx = \int\left(-\frac{1}{3}\cdot\frac{1}{x+2} + \frac{1}{3}\cdot\frac{1}{x-1}\right)dx$$
$$= -\frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{3}\int\frac{1}{x-1}dx$$
$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C \quad \#$$
$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$$

$\displaystyle\int \frac{1}{x+2}dx$

ให้ $u = x+2$; $du = dx$

$\displaystyle\therefore \int \frac{1}{u}du = \ln|u| + C$

$= \ln|x+2| + C$

Example 6.17 Evaluate $\displaystyle\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx.$

$\deg(p(x)) = 2$

$\deg(Q(x)) = 3$

สมมติ $\displaystyle\frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$

เอา $(x+1)(x-2)^2$ คูณตลอด

$$2x^2-3x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$
$$2x^2-3x+4 = A(x^2-4x+4) + B(x^2-x-2) + C(x+1)$$
$$2x^2-3x+4 = (Ax^2-4Ax+4A) + (Bx^2-Bx-2B) + (Cx+C)$$

Group เทอม:

$$2x^2-3x+4 = (A+B)x^2 + (-4A-B+C)x + (4A-2B+C)$$

เทียบ สปส.

$2 = A+B$ —①

$-3 = -4A-B+C$ —②

$4 = 4A-2B+C$ —③

③−②; $7 = 8A-B$ —④

①+④; $9 = 9A \Rightarrow A = 1$

$\therefore B = 1 \Rightarrow C = 2$

$$\therefore \frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

$$\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx = \int\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}\right)dx$$
$$= \int \frac{1}{x+1}dx + \int \frac{1}{x-2}dx + 2\int \frac{1}{(x-2)^2}dx$$
$$= \ln|x+1| + \ln|x-2| - \frac{2}{x-2} + C$$

Quadratic factor rule : For each factor of the form $(ax^2+bx+c)^m$ with $b^2-4ac<0$, the partial fraction decomposition contains the following sum of m partial fractions:
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_mx+B_m}{(ax^2+bx+c)^m},$$ where A_i, B_i $(i = 1, 2, \ldots, m)$ are constants.

Example 6.18 Evaluate $\displaystyle\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx.$

สมมติ $\displaystyle\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

คูณ $(x+1)(x^2+1)$; $3x^2+x-2 = A(x^2+1) + (Bx+C)(x-1)$

$$3x^2+x-2 = Ax^2+A+Bx^2-Bx+Cx-C$$
$$3x^2+x-2 = (A+B)x^2 + (-B+C)x + (A-C)$$

เทียบ สปส.

$3 = A+B$ —①

$1 = -B+C$ —②

$-2 = A-C$ —③

①+②; $4 = A+C$ —④

③+④; $2 = 2A \Rightarrow A = 1$

$B = 2$

$C = 3$

$$\therefore \frac{3x^2+x-2}{(x+1)(x^2+1)} = \frac{1}{x-1} + \frac{2x+3}{x^2+1}$$

$$\therefore \int \frac{3x^2+x-2}{(x+1)(x^2+1)}dx = \int \frac{1}{x-1}dx + \int \frac{2x+3}{x^2+1}dx$$

$$= \int \frac{1}{x-1}dx + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx$$

$\ln|x-1|$ $\qquad$ $3\arctan x$

ให้ $u = x^2+1$

$du = 2x\,dx$

$\therefore \int \frac{2x}{x^2+1}dx = \int \frac{1}{u}du = \ln|u|$

$= \ln|x^2+1|$

$$= \ln|x-1| + \ln|x^2+1| + 3\arctan x + C$$

Example 6.19 Evaluate $\int \frac{x+4}{x^2(x^2+4)}\,dx$.

สมมติ $\frac{x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$

ค่าที่ถูก $A = \frac{1}{4}$, $B = 1$, $C = -\frac{1}{4}$, $D = -1$

Hint $\int \frac{x+4}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$

Example 6.20 Evaluate $\int \frac{x^3-4x}{(x^2+1)^2}\,dx$.

สมมติ $\frac{x^3-4x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

ค่าที่ถูก

$A = 1$

$B = 0$

$C = -5$

$D = 0$

6.5.1 Integrating Improper Rational Functions

Example 6.21 Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$.

The integrand can be expressed as

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln|\frac{x-1}{x+2}|+C.$$

Example 6.19 Evaluate $\displaystyle\int \frac{x+4}{x^2(x^2+4)}\,dx$.

สมมติ $\dfrac{x+4}{x^2(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+4}$

คูณด้วย $x^2(x^2+4)$ ทั้งสองข้าง;

$$\begin{aligned} x+4 &= Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2 \\ x+4 &= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Dx^2 \\ x+4 &= (A+C)x^3 + (B+D)x^2 + 4Ax + 4B \end{aligned}$$

เทียบ สปส. :

$$\left.\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ 4A &= 1 \\ 4B &= 4 \end{aligned}\right\} \quad A = \frac{1}{4},\ B = 1 \quad \therefore C = -\frac{1}{4},\ D = -1$$

$$\begin{aligned} \therefore \int \frac{x+4}{x^2(x^2+4)}dx &= \frac{1}{4}\int \frac{1}{x}dx + \int \frac{1}{x^2}dx + \int \frac{-\frac{1}{4}x - 1}{x^2+4}dx \\ &= \frac{1}{4}\int \frac{1}{x}dx + \int \frac{1}{x^2}dx - \frac{1}{4}\int \frac{x+4}{x^2+4}dx \\ &= \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{4}\left[\int \left(\frac{x}{x^2+4} + \frac{4}{x^2+4}\right)dx\right] \\ &= \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{8}\ln|x^2+4| - \arctan\left(\frac{x}{2}\right) + C \ \# \end{aligned}$$

พิจารณา $\displaystyle\int \frac{x}{x^2+4}dx$

ให้ $u = x^2+4$

$du = 2x\,dx$

$\displaystyle\therefore \int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+4| + C$

Example 6.20 Evaluate $\displaystyle\int \frac{x^3-4x}{(x^2+1)^2}\,dx$.

สมมติ $\dfrac{x^3-4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

คูณด้วย $(x^2+1)^2$ ทั้งสองข้าง

$$\begin{aligned} x^3-4x &= (Ax+B)(x^2+1) + (Cx+D) \\ x^3-4x &= Ax^3 + Ax + Bx^2 + B + Cx + D \\ x^3-4x &= Ax^3 + Bx^2 + (A+C)x + (B+D) \end{aligned}$$

เทียบ สปส:

$$\left.\begin{aligned} A &= 1 \\ B &= 0 \\ A + C &= -4 \\ B + D &= 0 \end{aligned}\right\} \quad C = -5,\ D = 0$$

$$\therefore \int \frac{x^3-4x}{(x^2+1)^2}dx = \boxed{\int \frac{x}{x^2+1}dx} - \boxed{\int \frac{5x}{(x^2+1)^2}dx}$$

$u = x^2+1$

$du = 2x\,dx$

$\displaystyle\therefore \int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{1}{u}du$

$= \frac{1}{2}\ln|x^2+1| + C_1$

$u = x^2+1$

$du = 2x\,dx$

$\displaystyle\therefore \int \frac{5x}{(x^2+1)^2}dx = \frac{5}{2}\int \frac{1}{u^2}du$

$= \frac{5}{2}\left(-\frac{1}{u}\right) + C = -\frac{5}{2}\cdot\frac{1}{x^2+1} + C$

$$= \frac{1}{2}\ln|x^2+1| - \frac{5}{2}\cdot\frac{1}{x^2+1} + C \ \#$$