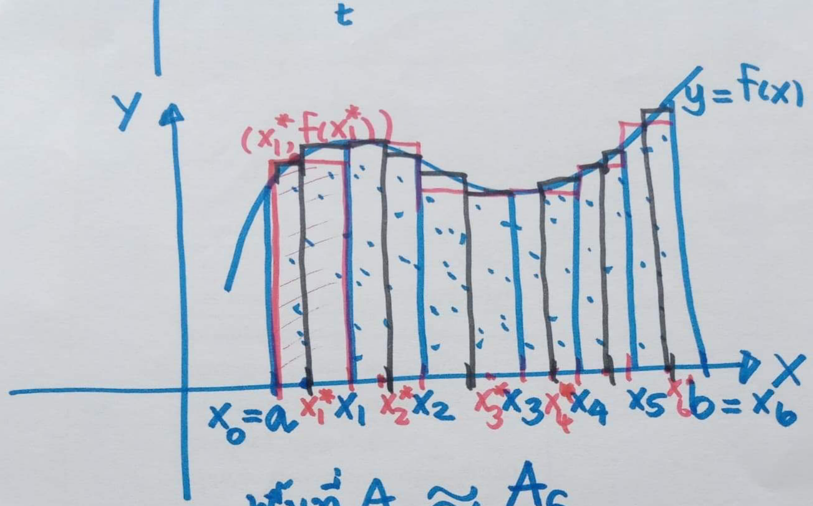
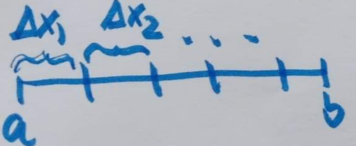


พื้นที่ A $\approx A_6$

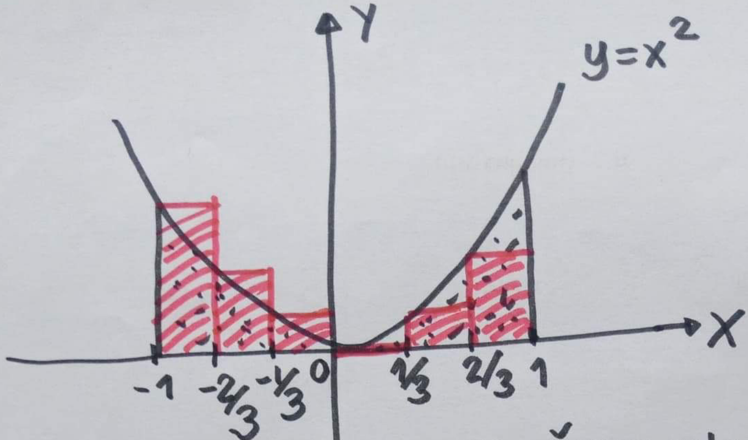
$$= f(x_1^*)(x_1-x_0) + f(x_2^*)(x_2-x_1) + f(x_3^*)(x_3-x_2) + f(x_4^*)(x_4-x_3) + f(x_5^*)(x_5-x_4) + f(x_6^*)(x_6-x_5)$$



$$= \sum_{k=1}^{6} f(x_k^*)\underbrace{(x_k - x_{k-1})}_{\Delta x_k}$$

$$\text{พื้นที่ A} = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(x_k^*)\Delta x_k}_{\text{ผลรวมของ } \square}$$

Ex



* ต้องการประมาณพื้นที่ใต้กราฟ $y=x^2$ บน $[-1,1]$

① แบ่ง $[-1,1]$ ออกเป็น 6 ช่วงย่อย

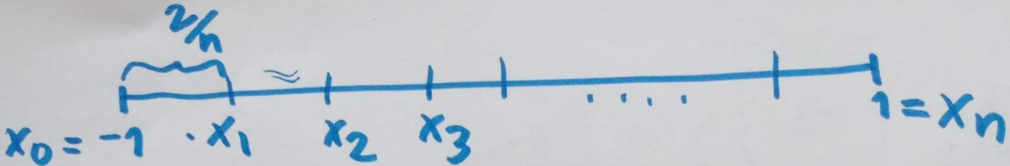
ความกว้างของช่วงย่อย (width) $= \frac{2}{6} = \frac{1}{3}$

② $$A_6 = (-1)^2\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)^2\left(\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2\left(\frac{1}{3}\right) + (0)^2\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$$

③ แบ่ง $[-1,1]$ ออกเป็น n ช่วงย่อย

ความกว้างของช่วงย่อย $= 2/n$

④ $$A_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$



$$x_1 = -1 + \frac{2}{n}$$

$$x_2 = -1 + 2\left(\frac{2}{n}\right)$$

$$x_3 = -1 + 3\left(\frac{2}{n}\right)$$

$$\vdots$$

$$x_k = -1 + k\left(\frac{2}{n}\right)$$

$$A_n = \sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k$$

$$= \sum_{k=1}^{n} (x_{k-1})^2\,\Delta x_k$$

$$= \sum_{k=1}^{n} \left(-1 + (k-1)\left(\frac{2}{n}\right)\right)^2 \left(\frac{2}{n}\right)$$

#

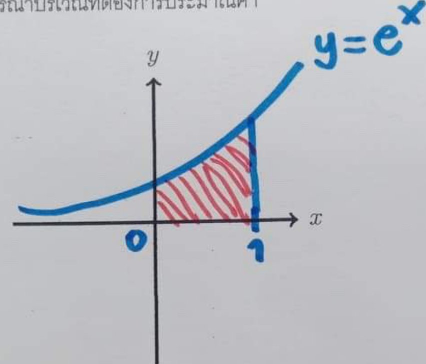
ครั้งที่ 21: แบบฝึกหัด

สำหรับเช็คชื่อ ประจำวันศุกร์ที่ 25 ตุลาคม พ.ศ.2562 ทำพร้อมกันในห้องเรียน

ชื่อ-สกุล.. รหัสนักศึกษา.............................. ลำดับที่............

ต้องการหาพื้นที่ที่อยู่ใต้เส้นโค้ง $f(x) = e^x$ บนช่วง $[0, 1]$ โดยการประมาณด้วยรูปสี่เหลี่ยมผืนผ้า

1. ให้นักศึกษาวาดกราฟ พร้อมพิจารณาบริเวณที่ต้องการประมาณค่า



2. ให้ $m = 111$
สมมติว่าต้องการแบ่งช่วงปิด $[0, 1]$ ให้เท่ากัน m ช่วง จะได้ว่า แต่ละช่วงย่อยจะมีความกว้าง $\Delta x =$ ....$1/111$

3. ถ้าสมมติให้ความสูงของสี่เหลี่ยมผืนผ้า พิจารณาจากค่าของฟังก์ชันทางด้านซ้ายมือของช่วง ดังนั้น พื้นที่โดยประมาณจะเขียนได้เป็น (ไม่ต้องคำนวณค่าฟังก์ชัน ให้ตอบติดในรูป $f(0 + km)$)

$$A_m = e^0\left(\frac{1}{m}\right) + e^{1/m}\left(\frac{1}{m}\right) + e^{2/m}\left(\frac{1}{m}\right) + \dots + e^{(m-1)/m}\left(\frac{1}{m}\right)$$

4. จากข้อ 4. สามารถเขียนอยู่ในรูปแบบสัญลักษณ์แทนผลบวก (Summation) ได้เป็น

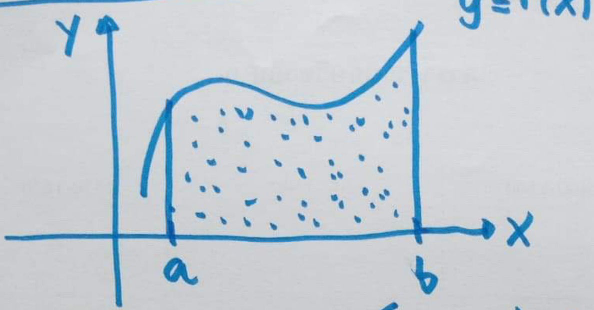
$$A_m = \sum_{k=1}^{m} e^{(k-1)/m}\left(\frac{1}{m}\right) = \sum_{k=0}^{m-1} e^{k/m}\left(\frac{1}{m}\right)$$

5. หากแบ่งเป็น n ช่วงย่อย พื้นที่ A ภายใต้เส้นโค้ง $f(x) = e^x$ บนช่วงปิด $[0, 1]$ จะหาได้โดยใช้ลิมิต โดย

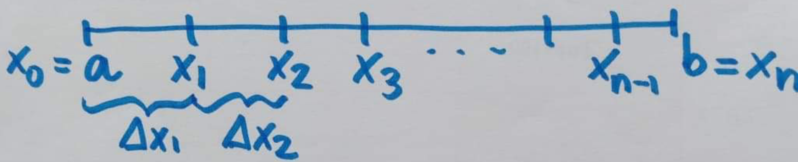
$$A = \lim_{m\to\infty} A_m$$

$$= \lim_{m\to\infty} \sum_{k=1}^{m} e^{(k-1)/m}\left(\frac{1}{m}\right)$$

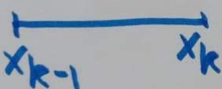
Definite Integral



① แบ่ง [a,b] ออกเป็น n ช่วงย่อย

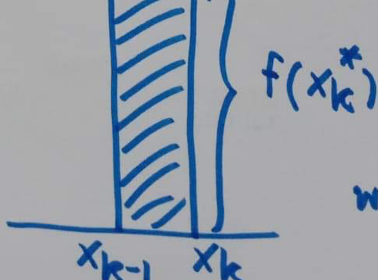


$$\Delta x_1 = x_1 - x_0$$
$$\Delta x_2 = x_2 - x_1$$
$$\Delta x_3 = x_3 - x_2$$
$$\vdots$$
$$\Delta x_n = x_n - x_{n-1}$$



$$\Delta x_k = x_k - x_{k-1}$$

② เลือก $x_k^* \in [x_{k-1}, x_k]$



พ.ท. $= f(x_k^*)\Delta x_k$

③ รวมพ.ท.

$$A_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

④ พ.ท. ใต้เส้นโค้ง $y=f(x)$ บน $[a,b]$

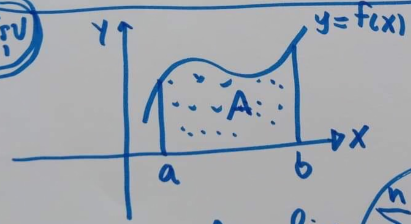
$$A = \lim_{n\to\infty} A_n$$

$$= \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k}_{\substack{f \text{ ต่อเนื่อง} \\ \text{บน } [a,b] \\ \Rightarrow \text{ ลิมิตหาค่าได้}}}$$

$$\Big\|$$

แทนด้วย $\int_a^b f(x)\,dx$

สรุป



Riemann sum

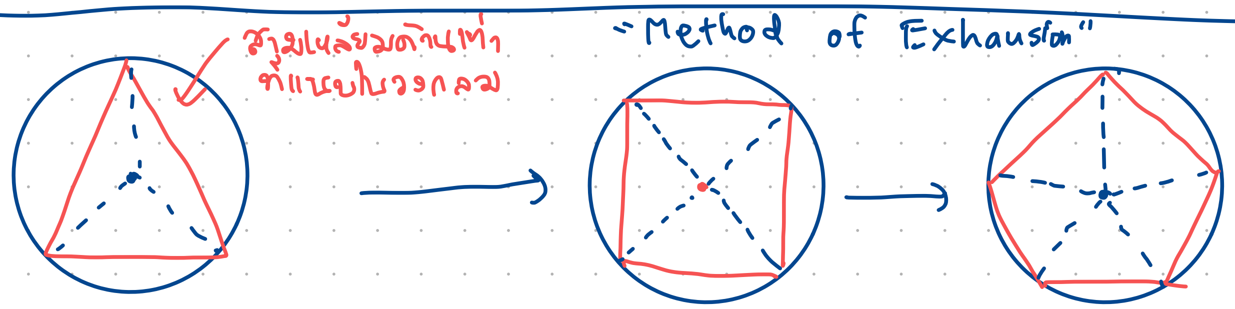
$$\text{พ.ท. } A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$$= \int_a^b f(x)\,dx$$

Indefinite integral $\longrightarrow$

- u-substitution
- by parts
- trigonometric function
- trigonometric substitution
- partial fraction

"Method of Exhaustion"



ให้ $A(n)$ คือ พ.ท. ของรูป n เหลี่ยมด้านเท่ามุมเท่า (regular n-gon) แนบในวงกลม รัศมี 1 หน่วย

$$\lim_{n\to\infty} A(n) = \pi$$

7

Definite Integration and its Applications

7.1 An Overview of Area Problem

Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 7.1).

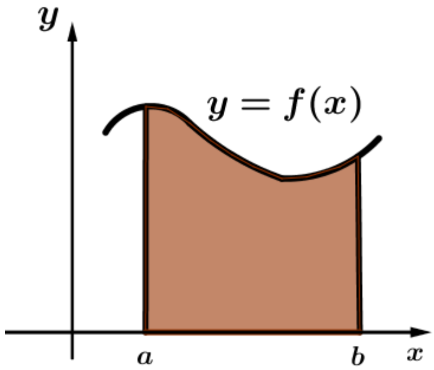


Figure 7.1: Area problem

7.2 The Definition of Area as a Limit; Sigma Notation

7.2.1 Sigma Notation

ขีดจำกัด ของผลบวก →

index →

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \dots + f(n)$$

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called sigma notation or summation notation because it uses the uppercase Greek letter $\sum$ to denote various kinds of sums.

If $f(k)$ is a function of k, and if m and n are integers such that $m \leq n$, then

$$\sum_{k=m}^{n} f(k)$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$.

Example 7.1

$\sum_{k=4}^{8} k^3 =$ $4^3 + 5^3 + 6^3 + 7^3 + 8^3$

$\sum_{k=0}^{5} (-1)^k (2k-1) =$ $(-1)^0(2(0)-1) + (-1)^1(2(1)-1) + (-1)^2(2(2)-1) + (-1)^3(2(3)-1) + (-1)^4(2(4)-1) + (-1)^5(2(5)-1)$

$= -1 - 1 + 3 - 5 + 7 - 9$

7.2.2 Properties of Sums

Theorem 7.1

(a) $\sum_{k=1}^{n} \underline{c}a_k = c \sum_{k=1}^{n} a_k$

(b) $\sum_{k=1}^{n} (\underline{a_k} + \underline{b_k}) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(Gauss)

$\circledast$ $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

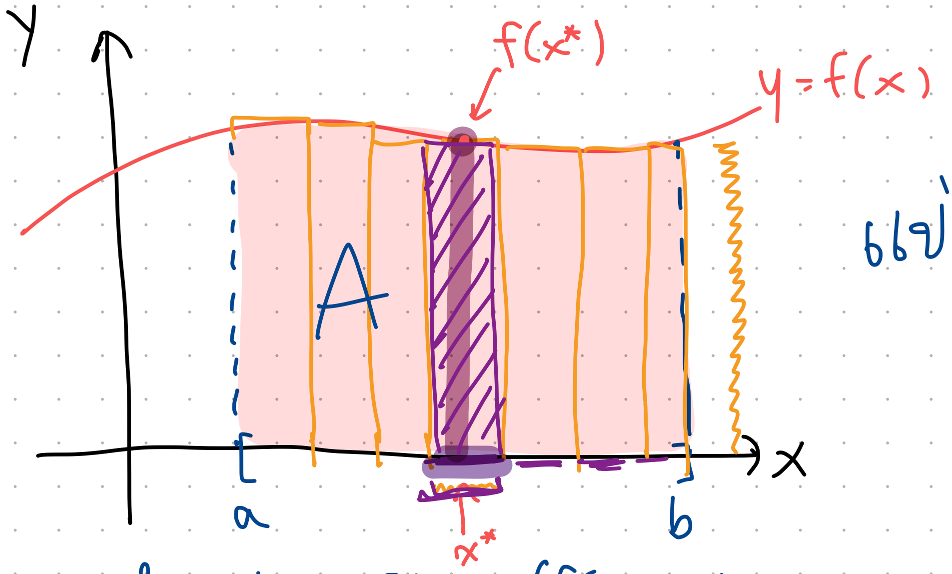
$\circledast$ $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7.2.3 The Rectangle Method for Finding Areas

One approach to the area problem is to use Archimedes method of exhaustion in the following way:

Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter – it can be above the center, above an endpoint, or above any other point in the subinterval.

For each n, the total area of the rectangles can be viewed as an approximation to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 7.2).



แบ่ง $\boxed{\text{ผ}}$ n ชิ้น

ต้องการหา พ.ท. ใต้กราฟ $y=f(x)$ $(f(x) \geq 0)$ บน $[a,b]$

↳ แบ่ง พ.ท. ออกเป็นส่วนย่อยๆ ⇒ เป็นรูป $\boxed{\text{ผ}}$ ⇒ ความยาวฐาน, สูง

ต้องหาความกว้างฐาน ⇒ แบ่ง $[a,b]$ ออกเป็นช่วงย่อยๆ ⇒ ความยาวของช่วงย่อย

หาความสูงของ $\boxed{\text{ผ}}$ ⇒ ในช่วงย่อย เลือกตัวแทนมาสักตัว (x^* ในช่วงย่อย)

↳ ความสูงคือ $f(x^*)$

ให้ A_n เป็นผลรวมของรูป $\boxed{\text{ผ}}$ n ชิ้น ⇒ $\boxed{\lim_{n \to \infty} A_n = A}$

That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n\to\infty} A_n$$

We will call this the rectangle method for computing A.

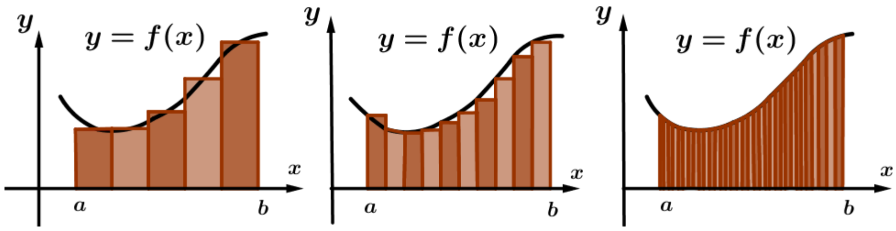


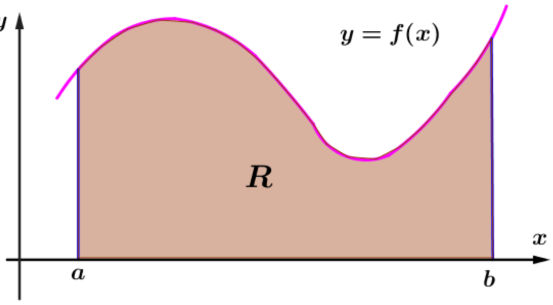
Figure 7.2: Finding Area

7.2.4 A Definition of Area

DEFINITION 5.1 (Area Under a Curve) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

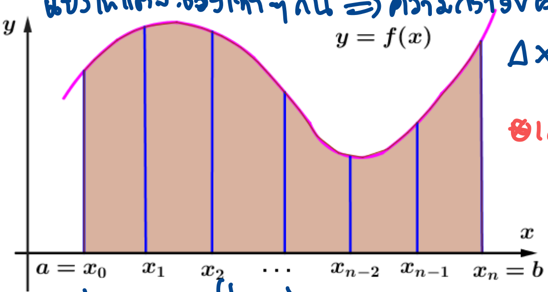
① ตีตารางประมาณ บริเวณ R ด้วย ▭ n ช่อง



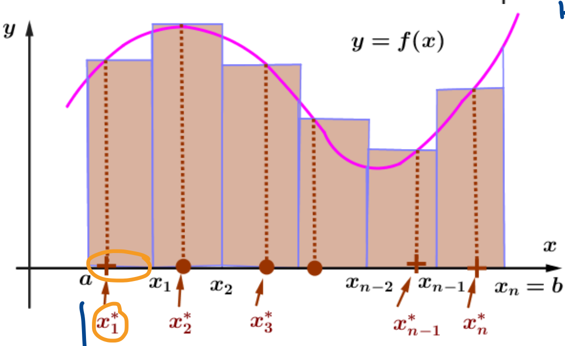
② แบ่งช่วงปิด [a,b] ออกเป็น n ช่วงย่อย แบ่งให้แต่ละช่วงเท่าๆ กัน ⇒ ความกว้างของช่วงย่อย

$\Delta x := \dfrac{b-a}{n}$

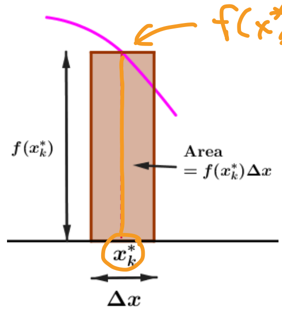
* เลือกความกว้าง ▭



$(b-a)$



$\Delta x + \Delta x + \quad \Delta x \quad \Delta x \quad \Delta x$



$f(x_*^*)$

③ เลือกตัวแทนในช่วงย่อย (x_i^*) แล้ว ความสูงของ ▭ นั้นคือ $f(x_i^*)$

$\therefore$ พ.ท. ▭ $= \underbrace{f(x_i^*)}_{\text{ความสูง}} \underbrace{\Delta x}_{\text{ความกว้าง}}$

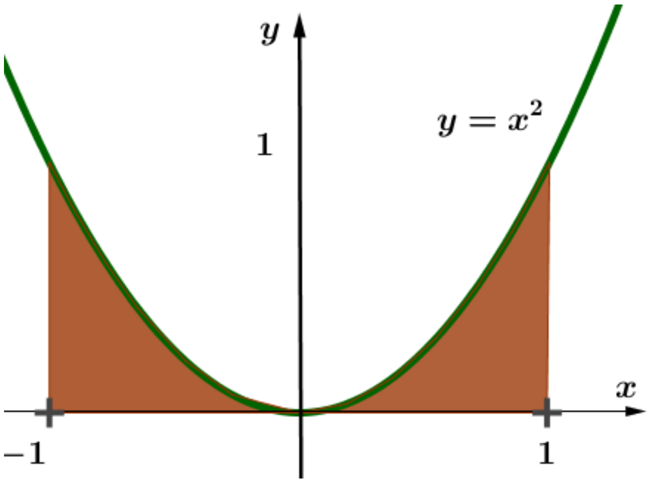
$\therefore A_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$

$= \sum_{i=1}^{n} f(x_i^*)\Delta x$

Take lim. $(n \to \infty) \Rightarrow A$

$A = \lim_{n\to\infty} A_n = \lim \sum_{i=1}^{n} f(x_i^*)\Delta x$

It is probably easiest to see how we do this with an example. So let's determine the area between $f(x) = x^2$ on $[-1, 1]$. In other words, we want to determine the area of the shaded region below.

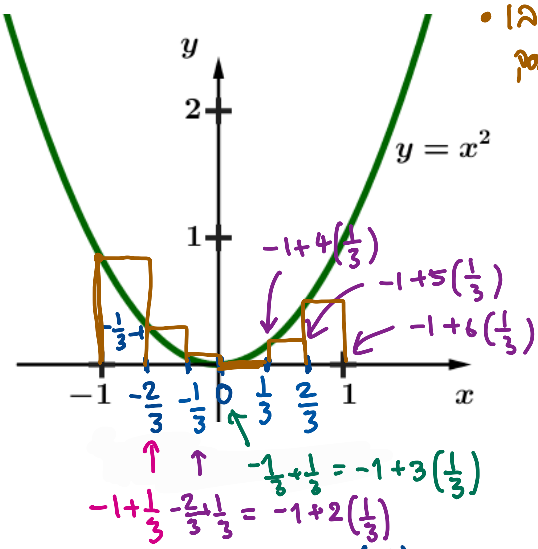


① แบ่ง $[-1, 1]$ ออกเป็น n ช่วงย่อย
② เลือกความสูง ของแต่ละ ▭ ย่อยๆ
③ Sum ผลบวก ▭

Figure 7.3: $y = x^2$

So, let's divide up the interval into 6 subintervals and use the function value on the left of each interval to define the height of the rectangle.

• เลือกความสูงจาก จุดปลายด้านซ้ายของ



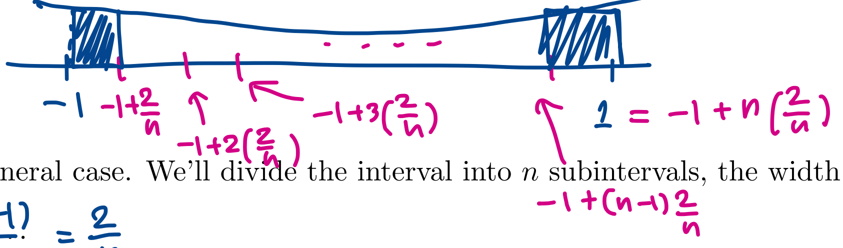
First, the width of each of the rectangles is .... $\Delta x = \frac{1-(-1)}{6} = \frac{2}{6} = \frac{1}{3}$

The height of each rectangle is determined by the function value on the left. Here is the estimated area.

$$A_6 = f(-1)\left(\tfrac{1}{3}\right) + f\left(-\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) + f\left(-\tfrac{1}{3}\right)\tfrac{1}{3} + f(0)\tfrac{1}{3} + f\left(\tfrac{1}{3}\right)\tfrac{1}{3} + f\left(\tfrac{2}{3}\right)\tfrac{1}{3}$$

$$= (-1)^2\left(\tfrac{1}{3}\right) + \left(-\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right) + \left(-\tfrac{1}{3}\right)^2\left(\tfrac{1}{3}\right) + 0^2\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{3}\right)^2\tfrac{1}{3} + \left(\tfrac{2}{3}\right)^2\tfrac{1}{3}$$

$$= \sum_{k=0}^{5} \underbrace{f\left(-1 + k\left(\tfrac{1}{3}\right)\right)}_{\text{ความสูง } \square} \underbrace{\tfrac{1}{3}}_{\text{ความกว้าง } \square}$$



Now, let's move on to the general case. We'll divide the interval into n subintervals, the width of each of the rectangles is ...$\frac{1-(-1)}{n} = \frac{2}{n}$

The total area A_n of the n rectangles will be

$$A_n = f(-1)\frac{2}{n} + f\left(-1+\frac{2}{n}\right)\frac{2}{n} + f\left(-1+2\cdot\frac{2}{n}\right)\frac{2}{n} + \dots + f\left(-1+(n-1)\frac{2}{n}\right)\cdot\frac{2}{n} \tag{7.1}$$

$$= \sum_{k=0}^{n-1} \boxed{f\left(-1+k\cdot\frac{2}{n}\right)}\,\frac{2}{n} \longrightarrow f(x) = x^2$$

$$A_n = \sum_{k=0}^{n-1}\left(-1+k\cdot\frac{2}{n}\right)^2\cdot\frac{2}{n}$$

Table 7.1 below shows the result of evaluating (7.1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to ....$\frac{2}{3}$.....

n	6	10	100	1,000	10,000
A_n	0.7	0.68	0.6668	0.666668	0.66666668

Table 7.1: estimation of area

So, increasing the number of rectangles improves the accuracy of the estimation as we would guess. Later in this chapter we will show that

$$\lim_{n\to\infty} A_n = \frac{2}{3}.$$

$$\therefore \lim_{n\to\infty} A_n = \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(-1+k\cdot\frac{2}{n}\right)^2\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{k=0}^{n-1}\left(-1+k\cdot\frac{2}{n}\right)^2$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{k=0}^{n-1}\left[1+2(-1)\left(k\cdot\frac{2}{n}\right)+\left(k\frac{2}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\sum_{k=0}^{n-1}\left[1-\frac{4k}{n}+\frac{4k^2}{n^2}\right]\right] = \lim_{n\to\infty}\left[\frac{2}{n}\left[\sum_{k=0}^{n-1}1-\frac{1}{n}\sum_{k=0}^{n-1}4k+\frac{1}{n^2}\sum_{k=0}^{n-1}4k^2\right]\right]$$

$$\vdots$$

$$= \frac{2}{3}$$

7.2.5 Net Signed Area

พื้นที่เครื่องหมายสุทธิ

*ถ้า $f(x) \geq 0$ เสมอ. $\Rightarrow \lim \Sigma$ — พ.ท. ใต้กราฟ.

If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

*ถ้า $f(x)$ มีทั้งบวกและลบ

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$\Rightarrow \lim \Sigma$ ~ เป็น พ.ท. net signed area

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas - the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the **net signed area**.

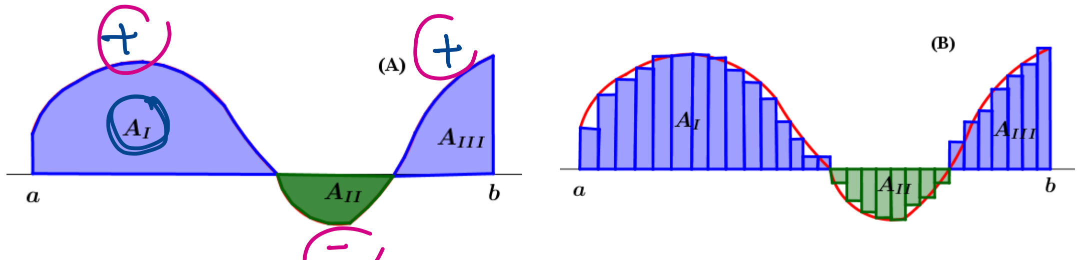


Figure 7.5: net sign area

For example, in Figure 7.5, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(AI + AIII) - AII = [\text{ area above } [a, b]] - [\text{ area below } [a, b]]$$

DEFINITION 5.2 (Net Signed Area) If the function f is continuous on $[a, b]$, then the net signed area A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

7.3 Definite Integral

7.3.1 Riemann Sums and the Definite Integral

In previous section, we assumed that for each positive number n, the interval $[a, b]$ was subdivided into n subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths; however, if we are to exhaust an area with rectangles of different widths, then it is important that successive subdivisions are constructed in such a way that the widths of all the rectangles approach zero as n increases (Figure 7.6-left). Thus, we must preclude the kind of situation that occurs in Figure 7.6-right in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as n increased.

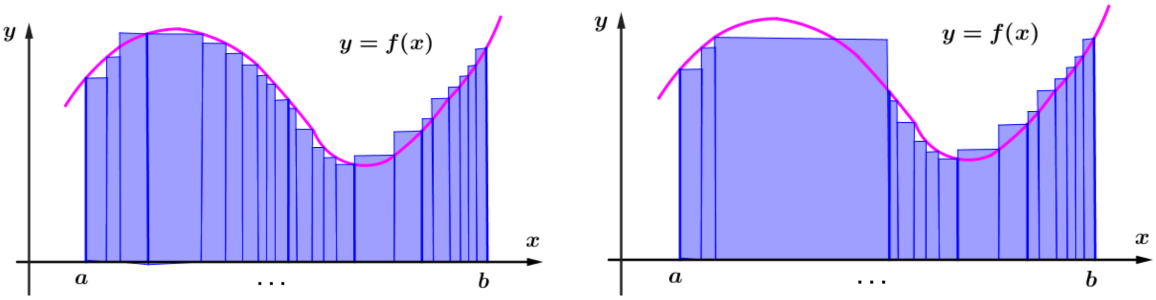
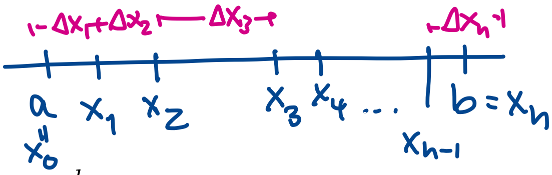


Figure 7.6: Definite integral แบ่ง [a,b] เป็น n ช่วง
แต่ละช่วงยาวไม่เท่ากัน



A partition of the interval $[a, b]$ is a collection of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n1} < x_n = b$$

that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, \Delta x_n = x_n - x_{n-1}$$

$$(\Delta x_k = x_k - x_{k-1})$$

The partition is said to be regular provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}.$$

For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the size of these widths. One approach is to let $\max \Delta x_k$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_k$ is called the **mesh size** of the partition. For example, Figure 7.7 shows a partition of the interval $[0, 6]$ into four subintervals with

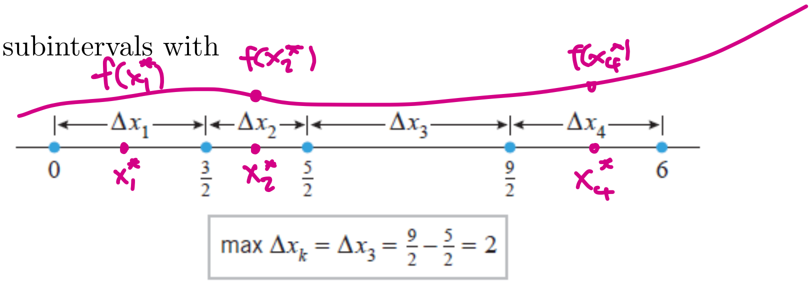


Figure 7.7: partition of [0,6]

a Δx_k b

If we are to generalize Definition 7.2.4 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

Δx ยาวไม่เท่ากัน

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \text{ is replaced by } \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

เดิม Take lim: $\lim_{n\to\infty} A_n$

ใหม่ $\lim_{\max \Delta x \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

We also need to replace the expression $n \to \infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose.

รวม พ.ท.ที่หมดใต้กราฟ $\Rightarrow$ พ.ท.บน $[a, b]$.

DEFINITION A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

ยิ่งให้ช่วงที่กว้างที่สุดเล็กที่สุด $\Updownarrow$ แบ่ง จน.ชิ้นเยอะๆ

ความสูงของช่วงย่อย

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

ความกว้างช่วงย่อย

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

ขีดจำกัดบน upper limit

integrand

ขีดจำกัดล่าง lower limit

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

ความหมาย $\Rightarrow$ net signed area บน $[a, b]$

"Definite integral : อินทิกรัลจำกัดเขต"

which is called the *definite integral* of f from a to b. The numbers a and b are called the *lower limit of integration* and the *upper limit of integration* , respectively, and $f(x)$ is called the *integrand* .

Theorem 7.2 If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x)dx.$$

$\int f(x) dx$

Fundamental Theorem of Calculus

Example 7.2 Use the areas shown in the figure to find

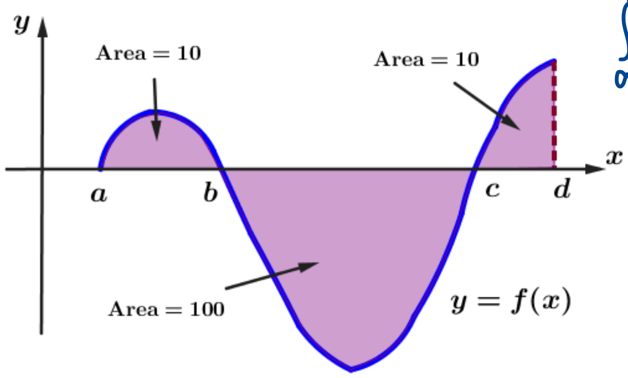
$\int_a^b f(x)dx$ ความหมาย = net signed area

(a) $\int_a^b f(x)dx = 10$ (b) $\int_b^c f(x)dx = -100$ (c) $\int_a^c f(x)dx = -90$ (d) $\int_a^d f(x)dx = -80$

Solution

$\int_a^c f(x)dx = 10 + (-100) = -90$

$\int_a^d f(x)dx = 10 + (-100) + 10 = -80$



Example 7.3 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2\,dx$ (b) $\int_0^1 \sqrt{1-x^2}\,dx$