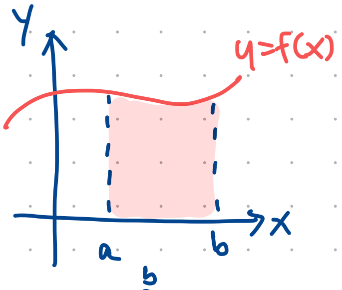
ทบทวน



① พ.ท. $A = \int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

② $A(x)$: ฟังก์ชันพ.ท.ใต้กราฟ จาก a ถึง x

① $A'(x) = f(x)$
② $A(0) = 0$
③ $A(b) = A$

$F(x) = A(x) + C$
($F(x)$ เป็นปฏิยานุพันธ์ของ $f(x)$)

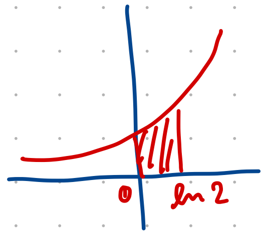
$\boxed{\text{FTC}}$

$\left(\int f(x)\,dx = F(x) + C \right)$

Part 1 f cont บน I , $F(x)$ เป็นปฏิยานุพันธ์ของ f

$$\therefore \boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

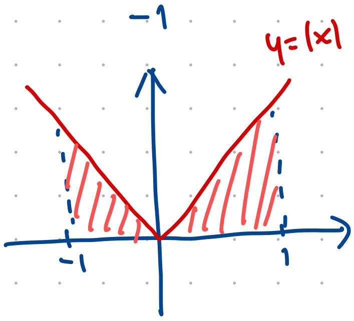
Ex $\int_0^{\ln 2} e^x\,dx$



พิจารณา $\int e^x\,dx = e^x + C$

$$\therefore \int_0^{\ln 2} e^x\,dx = e^x \Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$$

Ex $\int_{-1}^{1} |x|\,dx$



พิจารณา $f(x) = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$ เป็น piecewise-defined f^n

$$\therefore \int_{-1}^{1} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_0^1 x\,dx$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1$$

$$= \left[-\cancel{0} - \left(-\frac{(-1)^2}{2}\right)\right] + \left[\frac{1^2}{2} - \cancel{\frac{0^2}{2}}\right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

#

Ex Daily dose P.82 : $\int_{-2}^{1} g(t)\,dt$; $g(t) = \begin{cases} t^2+1 & ; t > -1 \\ \frac{2}{t} & ; t \leq -1 \end{cases}$

$$\therefore \int_{-2}^{1} g(t)\,dt = \int_{-2}^{-1} \frac{2}{t}\,dt + \int_{-1}^{1} (t^2+1)\,dt$$

$$= 2\ln|t|\Big|_{-2}^{-1} + \left(\frac{t^3}{3} + t\right)\Big|_{-1}^{1}$$

⋮

7.4.2 Relationship between Definite and Indefinite Integrals

Let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant, then

$$\int_a^b f(x)dx = [F(b) + C] - [F(a) - C] = F(b) - F(a)$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$\int_a^b f(x)dx = [F(x) + C]_a^b = \left[\int f(x)dx\right]_a^b$$

which relates the definite and indefinite integrals.

Example 7.9

(a) $\displaystyle\int_4^9 x^2\sqrt{x}\, dx =$

(b) $\displaystyle\int_0^{\ln 3} 5e^x\, dx =$

(c) $\int_{-1/2}^{1/2} \dfrac{1}{\sqrt{1-x^2}}\, dx =$

7.4.3 Part 2 of the Fundamental Theorem of Calculus

Theorem 7.8 If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f ; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

*f เป็นฟังก์ชันต่อเนื่องบนช่วง
ที่เอาอินทิเกรต

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

dummy variable

$\int f(t)dt = \int f(x)dx$

$F(2) = \int_a^2 f(t)dt$

Example 7.10 Find $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$

(เดิม) $\int_1^x t^3\,dt = \frac{t^4}{4} + c \Big|_1^x = \frac{x^4}{4} + \cancel{c} - \frac{1}{4} - \cancel{c}$

$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$

FTC 2: Check $f(t) = t^3$ cont บน $[1, x]$ $\forall x$

ใช้ FTC Part 2

$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$

$f(t) = t^3$

Ex $\frac{d}{dx}\int_1^x e^{t^2}\,dt$ $\quad f(t) = e^{t^2}$ cont บน $[1, x]$ $\forall x$

FTC #2: $\frac{d}{dx}\left[\int_1^x e^{t^2}\,dt\right] = e^{x^2}$

Ex $\frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt$; $f(t) = \frac{\sin t}{t}$ cont $[1, x]$

FTC #2: $\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right] = \frac{\sin x}{x}$

7.4.4 Evaluating Definite Integrals by Substitution

7.4.5 Two Methods for Making Substitutions in Definite Integrals

A definite integral of the form

$$\boxed{\int_a^b [f(u(x))u'(x)]dx,}$$

ทำ u-substitution

(แทน $u(x) = \_\_\_\_\_\_$)

$du = u'(x)dx$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1. First evaluate the indefinite integral

$$\int [f(u(x))u'(x)]dx$$

by substitution, and then use the relationship

$$\int_a^b [f(u(x))u'(x)]dx = \left[\int [f(u(x))u'(x)]dx\right]_a^b,$$

Method 2. Make the substitution directly in the definite integral, and then replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u(a)$ and $u(b)$. This produces a new definite integral

แทน $u = u(x)$

$du = u'(x)dx$

$$\int_{\underset{u(a)}{a}}^{\overset{u(b)}{b}} \underbrace{\boxed{f(u(x))}}_{f(u)}\underbrace{\boxed{u'(x)]dx}}_{du} = \boxed{\int_{u(a)}^{u(b)} [f(u)]du}$$

that is expressed entirely in terms of u.

Ex $\displaystyle\int_1^3 (2x+1)^2\,dx$

ให้ $u=2x+1$; $du = 2dx$

$dx = \frac{du}{2}$

$$\therefore \int_1^3 (2x+1)^2 dx = \cancel{\int_1^3 \frac{u^2}{2} du}$$

$$= \cancel{\frac{u^3}{6}\Big|_1^3}$$

$$= \cancel{\frac{(3)^3}{6} - \frac{1^3}{6}} \quad \#$$

$\displaystyle\int (2x+1)^2 dx$

$u = 2x+1$; $du = 2dx$

$dx = \frac{1}{2}du$

$$\therefore \int (2x+1)^2 dx = \int \frac{u^2}{2} du$$

$$= \frac{1}{2}\frac{u^3}{3} + C$$

$$= \frac{1}{6}(2x+1)^3 + C$$

<u>ข้อควรระวัง</u> เมื่อมีการเปลี่ยนตัวแปรการอินทิเกรต (u-substitution) ของปริพันธ์จำกัดเขต

ขอบเขตเดิม จะต้องเปลี่ยนใหม่ เป็นขอบเขตใหม่ ที่สอดคล้องกับการเปลี่ยนตัวแปร

<u>วิธีที่ 2</u> เปลี่ยนขอบเขตไปพร้อมกันกับการเปลี่ยนตัวแปร

$\displaystyle\int_{x=1}^{x=3} (2x+1)^2 dx$ ให้ $u = 2x+1 \Rightarrow du = 2dx$

ถ้า $x = 1$ แล้ว $u = 3$

$x = 3$ แล้ว $u = 7$

$$\int_{x=1}^{x=3} (2x+1)^2 dx = \int_{u=3}^{u=7} \frac{u^2}{2} du = \frac{u^3}{6}\Big|_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \quad \#$$

<u>วิธีที่ 1</u> ทำการอินทิเกรตแบบเปลี่ยนตัวแปรให้เสร็จก่อน แล้วแทนขอบ

$\displaystyle\int_1^3 (2x+1)^2 dx$ พิจารณา $\displaystyle\int (2x+1)^2 dx$ ให้ $u = 2x+1$; $du = 2dx$

$$\therefore \int (2x+1)^2 dx = \int \frac{u^2}{2} du = \frac{u^3}{6} + C = \frac{(2x+1)^3}{6} + C$$

$$\therefore \int_1^3 (2x+1)^2 dx = \frac{(2x+1)^6}{6}\Big|_1^3 = \frac{(2(3)+1)^3}{6} - \frac{(2(1)+1)^3}{6} \quad \#$$

<u>แบบ</u> : $f(0)=1$, $f(1)=3$, $f(2)=3$, $f(3)=5$.

$$\int_{x=1}^{x=3} \underbrace{(f(x))}_{y}{}^3 \underbrace{\boxed{f'(x)\,dx}}_{dy} = \int_{s}^{t} y^3 dy \qquad f(3) \to t,\quad f(1) \to s$$

แล้ว $s = $ <u>3</u>

$t = $ <u>5</u>

มีการเปลี่ยนตัวแปรการอินทิเกรต

ให้ $y = f(x)$

$dy = f'(x)dx$

$\Downarrow$

$s = f(1) = 3$, $t = f(3) = 5$

Example 7.11 Use the two methods above to evaluate $\int_2^0 x(x^2+5)^3 dx$

อินทิเกรต เปลี่ยนตัวแปรให้ครบ แล้วแทนขอบเขต

Solution by Method 1.

พิจารณา $\int x(x^2+5)^3 dx$; ให้ $u = x^2+5$; $du = 2x\,dx$

$$\therefore \int x(x^2+5)^3 dx = \int \cancel{x}(u^3)\frac{du}{2\cancel{x}} = \frac{1}{2}\int u^3 du = \frac{1}{2}\frac{u^4}{4} + C = \frac{1}{8}(x^2+5)^4 + C$$

$$\therefore \int_2^0 x(x^2+5)^3 dx = -\int_0^2 x(x^2+5)^3 dx = -\left[\frac{1}{8}(x^2+5)^4\right]_0^2 = -\frac{1}{8}(9)^4 + \frac{1}{8}(5)^4$$

$$= -\frac{5936}{8}$$

เปลี่ยนขอบเขตไปพร้อมๆ กัน

Solution by Method 2.

ให้ $u = x^2+5$

$x = 2 \Rightarrow u = 9$

$x = 0 \Rightarrow u = 5$

$$\int_{x=2}^{x=0} x(x^2+5)^3 dx = \int_{u=9}^{u=5} \frac{u^3}{2} du = \frac{u^4}{8}\Big|_9^5 = \frac{5^4}{8} - \frac{9^4}{8} = -\frac{5936}{8}$$

$$= -\int_5^9 \frac{u^3}{2} du$$

Example 7.12 Evaluate

(a) $\int_0^{3/4} \frac{1}{1-x} dx$

Method 1 พิจารณา $\int \frac{1}{1-x} dx$

$u = 1-x$; $du = -dx$

$$\therefore \int \frac{1}{1-x} dx = -\int \frac{1}{u} du = -\ln|u| + C$$
$$= -\ln|1-x| + C$$

$$\therefore \int_0^{3/4} \frac{1}{1-x} dx = -\ln|1-x|\Big|_0^{3/4}$$
$$= -\ln\left|1-\frac{3}{4}\right| + \cancel{\ln|1-0|}^{\,0}$$
$$= -\ln\left|\frac{1}{4}\right| = -\ln 4^{-1} = \ln 4$$

Method 2 $u = 1-x$; $x = 0 \Rightarrow u = 1$

$x = 3/4 \Rightarrow u = \frac{1}{4}$

$$\therefore \int_{x=0}^{x=3/4} \frac{1}{1-x} dx = -\int_{u=1}^{u=1/4} \frac{1}{u} du = -\ln|u|\Big|_1^{1/4}$$
$$= -\ln\left|\frac{1}{4}\right| + \cancel{\ln|1|}^{\,0}$$
$$= \ln 4$$

(b) $\int_0^{\ln 3} e^x(1+e^x)^{1/2} dx$

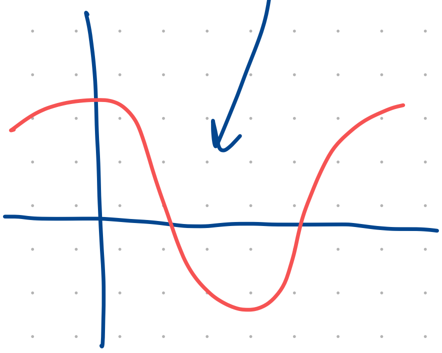
Method 2 $u = 1+e^x$; $du = e^x dx$

$x = 0 \Rightarrow u = 2$

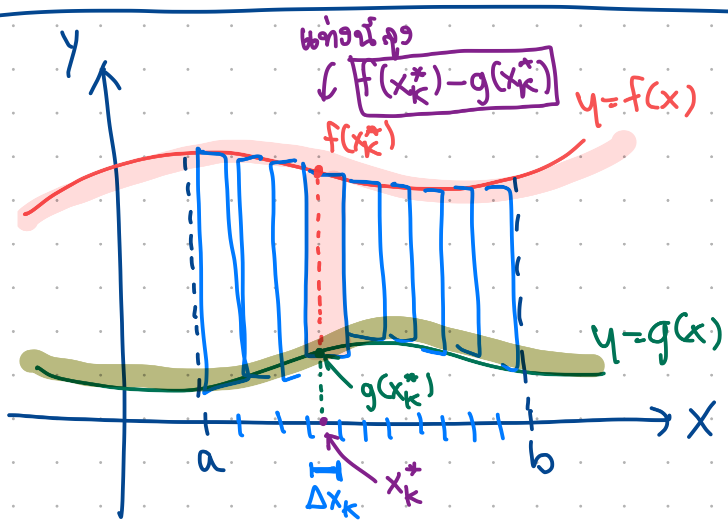
$x = \ln 3 \Rightarrow u = 1 + e^{\ln 3} = 1+3 = 4$

$$\therefore \int_0^{\ln 3} e^x(\underbrace{1+e^x}_{u})^{1/2} \underbrace{dx}_{du} = \int_2^4 u^{1/2} du = \frac{2u^{\frac{3}{2}}}{3}\Big|_2^4$$
$$= \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(2)^{\frac{3}{2}}$$
$$= \frac{2}{3}(8) - \frac{2}{3}(\sqrt{2})^3 \quad \#$$

$\int_a^b f(x)\,dx \rightarrow$ net-signed area



เราต้องการหา พ.ท. ที่ปิดล้อมด้วยเส้นโค้ง



$f(x) \geq g(x)$
สำหรับทุก $x \in [a,b]$

พ.ท. 1 แท่ง $= [f(x_k^*)-g(x_k^*)]\Delta x_k$

ผลรวม n แท่ง $= \sum_{k=1}^{n} [f(x_k^*)-g(x_k^*)]\Delta x_k$

พ.ท. แท้จริง $= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*)-g(x_k^*)]\Delta x_k$

$= \int_a^b [f(x)-g(x)]\,dx$

- พื้นที่
- ปริมาตรทรงตันที่เกิดจากการหมุนรอบแกน ← disk method, washer method, shell method
- Improper integral

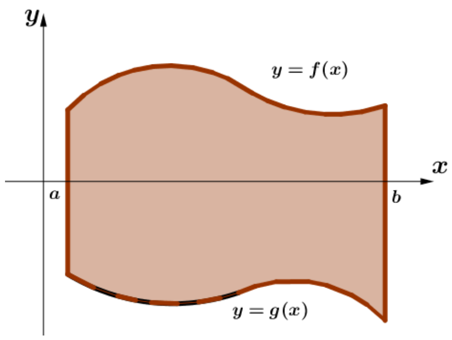
8

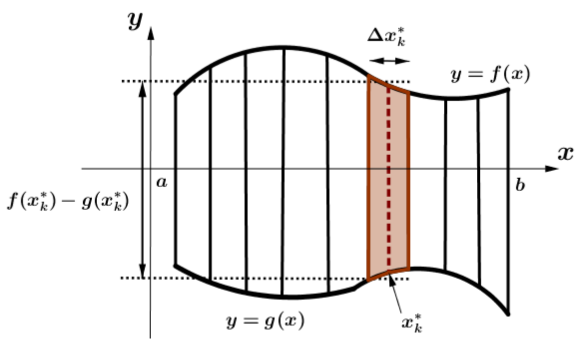
Applications of the Definite Integral in Geometry

8.1 Area between Two Curves

Theorem 8.1 If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$. Then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

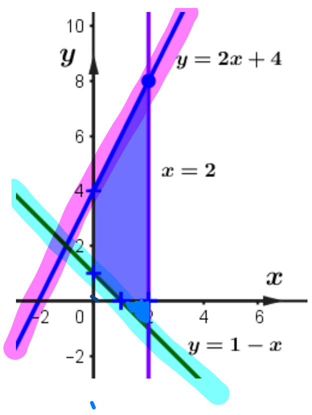
$$A = \int_a^b [f(x) - g(x)]dx. \tag{8.1}$$





$$A = \lim_{max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\, \Delta x_k = \int_a^b [f(x) - g(x)]dx$$

Example 8.1 Find the area of the region bounded above by $y = 2x + 4$, bounded below by $y = 1 - x$, and bounded on the sides by the line $x = 0$ and $x = 2$.



$$\therefore\ A = \int_0^2 \left([2x+4]-[1-x]\right)dx$$

$$= \int_0^2 (2x+4-1+x)\,dx$$

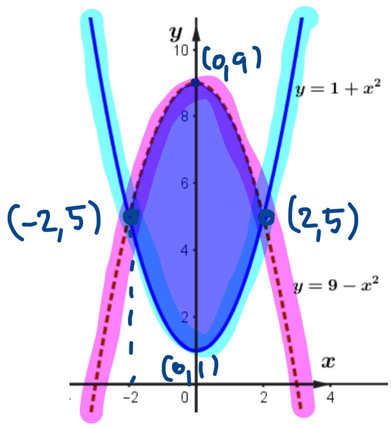
$$= \int_0^2 (3x+3)\,dx$$

$$= 3\int_0^2 (x+1)\,dx$$

$$= 3\left[\frac{x^2}{2}+x\right]_0^2$$

$$= 3\left[\left(\frac{4}{2}+2\right)-\left(\frac{0}{2}+0\right)\right] = 3(4) = 12 \text{ ตร.หน่วย } \#$$

Example 8.2 Find the area of the region enclosed by $y = 9 - x^2$ and $y = 1 + x^2$.



$$A = \int_{-2}^{2} \left([9-x^2]-[1+x^2]\right)dx$$

$$= \int_{-2}^{2} (8-2x^2)\,dx$$

⋮

ขอบเขตเกิดจากการตัดกันของ

$y = 9 - x^2$, $y = 1 + x^2$

$\Downarrow$

แก้ระบบสมการ

$$9 - x^2 = 1 + x^2$$

$$8 = 2x^2$$

$$\therefore\ x^2 = 4 \Rightarrow x = \pm 2$$

HW $= \dfrac{64}{3}$