ทบทวน ① ตรวจสอบการเป็นคำตอบ

② สมการ separable equation

$$h(y)\frac{dy}{dx} = g(x)$$

$$h(y)dy = g(x)dx \quad \leftarrow \text{differential form}$$

$$\int h(y)dy = \int g(x)dx$$

$$\vdots \qquad + C$$

Linear Equation $\quad \frac{dy}{dx} + p(x)y = q(x)$ $\quad$ สปส.ของ $\frac{dy}{dx}$ เป็น 1 เท่านั้น

	Original eq^{n}	Linear Form	$p(x)$	$q(x)$
①	$\frac{dy}{dx} + x^2y = e^x$	$\frac{dy}{dx} + x^2y = e^x$	x^2	e^x
②	$y' - 3e^xy = e^x$	$\frac{dy}{dx} - 3e^xy = e^x$	$-3e^x$	e^x
③	$x\frac{dy}{dx} = 1$ ↳ $\frac{dy}{dx} = \frac{1}{x}$	$\frac{dy}{dx} + 0y = \frac{1}{x}$	0	$\frac{1}{x}$
④	$xy' + 2y = 4x^2$	$y' + \frac{2}{x}y = 4x$	$\frac{2}{x}$	$4x$
⑤	$\frac{dy}{dx} + y - \frac{1}{1+e^x} = 0$	$\frac{dy}{dx} + y = \frac{1}{1+e^x}$	1	$\frac{1}{1+e^x}$

พิจารณา $\quad \frac{dy}{dx} + p(x)y = q(x) \quad$ — $(*)$

เป้าหมาย : พยายามทำให้พจน์ $(*)$ เกิดรูปที่เป็น separable ได้
เอา $I(x)$ $(*)$ แล้ว ฝั่งซ้ายมืออยู่ในรูป $\frac{d}{dx}[I(x)y]$ — ♡

(เดี๋ยวจะบอกว่า $I(x)$ คืออะไร)

เอา $I(x)$ คูณ $(*)$ ทั้งสมการ

$$I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x) \quad — (**)$$

พิจารณา $\quad \frac{d}{dx}[I(x)y] = I(x)\frac{dy}{dx} + y\frac{dI(x)}{dx} \quad$ — (♡♡)

(ต้องการแปลงรูป $(**)$ เป็น (♡♡))

พิจารณา

$$I(x)p(x)\cancel{y} = \frac{dI(x)}{dx}\cancel{y}$$

$$\frac{dI(x)}{dx} = I(x)p(x)$$

$$\frac{1}{I(x)}dI(x) = p(x)dx$$

$$\int \frac{1}{I(x)}dI(x) = \int p(x)dx$$

$$\ln|I(x)| = \int p(x)dx$$

$$\boxed{I(x) = e^{\int p(x)dx}}$$

ตัวประกอบการอินทิเกรต (Integrating factor)

9.4 Linear Equations

A first-order differential equation is called **linear** if it is expressible in the form

$$\frac{dy}{dx} + p(x) \cdot y = q(x). \tag{9.7}$$

Some examples of first-order linear differential equations are

$$\begin{aligned}
&\frac{dy}{dx} = x^3 - xy, \\
&\frac{dy}{dx} + x^2 y = e^x, \\
&\frac{dy}{dx} + (\sin x)y + x^3 = 0, \\
&\frac{dy}{dx} + 5y + 2 = 0.
\end{aligned}$$

We will assume that the functions $p(x)$ and $q(x)$ in (9.7) are continuous and we will look for a general solution that is valid on that interval. One method for doing this is based on the observation that if we define the function $I = I(x)$ by

$$I = e^{\int p(x)dx}. \tag{9.8}$$

then

$$\frac{dI}{dx} = e^{\int p(x)dx} \cdot \frac{d}{dx}\int p(x)dx = I \cdot p(x).$$

Thus,

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + \frac{dI}{dx}y = I\frac{dy}{dx} + Ip(x)y. \tag{9.9}$$

If (9.7) is multiplied through by I, it becomes

$$I\frac{dy}{dx} + Ip(x) \cdot y = Iq(x).$$

Combine this with (9.9), we have

$$\frac{d}{dx}(Iy) = Iq(x).$$

This equation can be solved for y by integrating both sides with respect to x and then dividing through by I to obtain

$$y = \frac{1}{I(x)} \int I(x)q(x)dx$$

which is a general solution of (9.7) on the interval. The function $I(x)$ in (9.8) is called **an integrating factor** for (9.7), and this method for finding a general solution of (9.7) is called **the method of integrating factors**.

The Method of Integrating Factors

Step 1 Calculate the integrating factor

$$I = e^{\int p(x)dx}.$$

Step 2 Multiply both sides of (9.7) by I and express the result as

$$\frac{d}{dx}(Iy) = Iq(x)$$

Step 3 Integrate both sides of the equation obtained in Step 2 and then solve for y. Be sure to include a constant of integration in this step.

Example 9.13 Find the general solution of

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$\frac{dy}{dx} - y = e^{2x}$$

$p(x) = -1$

$q(x) = e^{2x}$

(1) หา $I(x) = e^{\int p(x)dx} = e^{\int -1\,dx} = e^{-x}$

(2) เอา $I(x)$ คูณทั้งสมการ

$$I(x)\left[\frac{dy}{dx} - y\right] = I(x)e^{2x}$$

$$\frac{d}{dx}\left[I(x)y\right] = I(x)\cdot e^{2x}$$

$$\frac{d}{dx}(e^{-x}y) = e^{-x}\cdot e^{2x}$$

$$\frac{d}{dx}(e^{-x}y) = e^{x}$$

$$d(e^{-x}y) = e^{x}dx$$

$$\int d(e^{-x}y) = \int e^{x}dx$$

$$e^{-x}y = e^{x} + C$$

$$\boxed{y = e^{2x} + Ce^{x}}$$ #

$$\frac{dy}{dx} + p(x)y = q(x)$$

Example 9.14 Solve the initial value problem

$$x\frac{dy}{dx} - y = x, \quad y(1) = 2. \iff \boxed{\frac{dy}{dx} - \frac{1}{x}y = 1} \quad \text{---}(*)$$

$p(x) = -\frac{1}{x}$, $q(x) = 1$

(1) $I(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|} = \frac{1}{x}$; $x > 0$

(2) คูณ $(*)$ ด้วย $I(x)$

$$\frac{1}{x}\left[\frac{dy}{dx} - \frac{1}{x}y\right] = \frac{1}{x}\cdot 1$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{1}{x}$$

$$d\left(\frac{1}{x}y\right) = \frac{1}{x}dx$$

$$\int d\left(\frac{1}{x}y\right) = \int \frac{1}{x}dx$$

$$\frac{1}{x}y = \ln(x) + C \quad (x>0)$$

$$\boxed{y = x\ln x + Cx}$$

หา C ; $y(1) = 2$

$\therefore 2 = 1\ln 1 + C \Rightarrow C = 2$

$\therefore \boxed{y = x\ln x + 2x}$ #

Example 9.15 Find the general solution of

$$\frac{dy}{dx} = xe^x + y - 1$$

$$\frac{dy}{dx} - y = xe^x - 1 \quad \text{---}(*)$$

$p(x) = -1$, $q(x) = xe^x - 1$

(1) $I(x) = e^{\int p(x)dx} = e^{\int -1dx} = e^{-x}$

(2) $e^{-x}\left[\frac{dy}{dx} - y\right] = e^{-x}\left[xe^x - 1\right]$

$$\frac{d}{dx}\left[e^{-x}y\right] = x - e^{-x}$$

$$d\left[e^{-x}y\right] = (x - e^{-x})dx$$

$$\int d(e^{-x}y) = \int (x - e^{-x})dx$$

$$e^{-x}y = \frac{x^2}{2} + e^{-x} + C$$

$$\therefore \boxed{y = \frac{x^2e^x}{2} + 1 + Ce^x}$$ #

Example 9.16 Solve the initial value problem

$$\frac{dy}{dx} = \frac{x-1}{e^{2x}} - 2xy, \quad y(0) = 1.$$

$p(x) = 2x$; $q(x) = \frac{x-1}{e^{2x}}$

$$\frac{dy}{dx} + 2xy = \frac{x-1}{e^{2x}} \quad \text{---}(*)$$

(1) $I(x) = e^{\int 2xdx} = e^{x^2}$

(2) $e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = e^{x^2}\cdot\frac{(x-1)}{e^{2x}}$

$$\frac{d}{dx}\left[e^{x^2}y\right] = e^{x^2-2x}(x-1)$$

$$\therefore d(e^{x^2}y) = e^{x^2-2x}(x-1)dx$$

$$\int d(e^{x^2}y) = \int e^{x^2-2x}(x-1)dx$$

$$e^{x^2}y = \frac{1}{2}e^{x^2-2x} + C$$

$$\boxed{y = \frac{1}{2}e^{-2x} + Ce^{-x^2}}$$

พิจารณา $\int e^{x^2-2x}(x-1)dx$

ให้ $u = x^2 - 2x$; $du = 2(x-1)dx$

$\therefore \frac{1}{2}\int e^{\overset{u}{x^2-2x}}\,\overset{du}{2(x-1)dx} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2-2x} + C$

$y(0) = 1$

$\therefore 1 = \frac{1}{2}e^0 + Ce^{0^2}$

$1 = \frac{1}{2} + C$

$\therefore \boxed{C = \frac{1}{2}}$

คำตอบคือ $\boxed{y = \frac{1}{2}e^{-2x} + \frac{1}{2}e^{-x^2}}$

Example 9.17 Find the general solution of

$$\frac{dy}{dx} = \frac{cosx - y}{x}, \quad x > 0.$$

$\frac{dy}{dx} = \frac{\cos x}{x} - \frac{1}{x}y$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{\cos x}{x}$ $p(x) = \frac{1}{x}$ $q(x) = \frac{\cos x}{x}$

(1) $I(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

(2) $x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x \cdot \frac{\cos x}{x}$

$\frac{d}{dx}(xy) = \cos x$

$d(xy) = \cos x \, dx$

$\int d(xy) = \int \cos x \, dx$

$xy = \sin x + C$

$\therefore \boxed{y = \frac{\sin x}{x} + \frac{C}{x}}$

9.5 Applications of Differential Equations

9.5.1 Exponential Growth Law

$\boxed{\text{Growth + Decay Model}}$

In general, if the rate of change of a quantity Q with respect to time is proportional to the amount of Q present and $Q(0) = Q_0$, then, we obtain the following theorem:

Exponential Growth Law If $\frac{dQ}{dt} = rQ$ and $Q(0) = Q_0$ then $Q = Q_0e^{rt}$ where

- Q_0 is amount of Q at $t = 0$
- r is relative growth rate
- t is time
- Q is quantity at time t

If r is positive, this becomes **exponential growth**. If r is negative, this becomes an **exponential decay** problem.

The constant r in the exponential growth law is called ***the relative growth rate*** . If the relative growth rate is $r = 0.02$, then the quantity Q is growing at a rate $dQ/dt = 0.02Q$ (that is 2% of the quantity Q per unit of time t). Note the distinction between the relative growth rate r and the rate of growth dQ/dt of the quantity Q. Relative growth rate is 0.02 and the rate of growth is $0.02Q$. Once we know that the rate of growth of something is proportional to the amount present, we know that it has exponential growth and we can use the exponential growth formula.

Growth & Decay Model

สมมติว่า ปริมาณ Q ณ เวลา t คือปริมาณ $Q(t)$

อัตราการเปลี่ยนแปลงของ Q เทียบกับ เวลา คือ r

→ อัตราการเปลี่ยนแปลงของ Q เทียบกับเวลา เป็นสัดส่วนโดยตรงกับ จำนวน $Q(t)$

$$\Downarrow$$

$$\frac{dQ}{dt} \propto Q(t) \iff \boxed{\frac{dQ}{dt} = rQ(t)}$$

growth rate/ decay rate

ถ้า $r>0 \Rightarrow$ growth model

$r<0 \Rightarrow$ decay model

เริ่มต้น ที่ เวลา $t=0$ มีปริมาณเป็น Q_0

$$Q(0) = Q_0$$

$$\frac{dQ}{dt} = rQ$$

$$\frac{1}{Q}dQ = r\,dt$$

$$\int \frac{1}{Q}dQ = \int r\,dt$$

$$\ln|Q| = rt + c$$

$$Q(t) = e^{rt+c} = Ce^{rt} \quad (\because C = e^c)$$

$$\therefore \quad Q_0 = Q(0) = Ce^{r(0)} \Rightarrow C = Q_0$$

$$\therefore \boxed{Q(t) = Q_0e^{rt}}$$

Example 9.18 The world population passed 1 billion in 1804, 2 billion in 1927, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987, and 6 billion in 1999, as illustrated in Figure 9.1. Population growth over certain periods can be approximated by the exponential growth law.

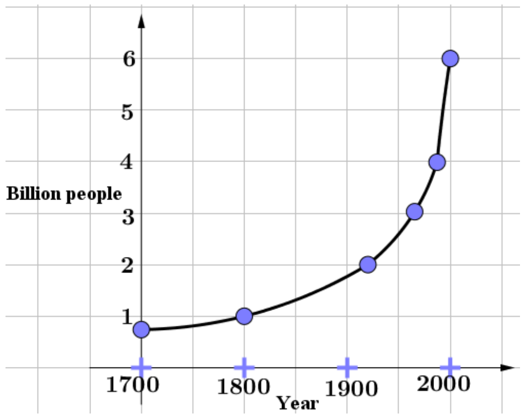


Figure 9.1: World population growth

Example 9.19 Population Growth India had a population of about 1.2 billion in 2010. Let P represent the population (in billions) t years after 2010, and assume a growth rate of 1.5% compounded continuously.

(A) Find an equation that represents India's population growth after 2010, assuming that the 1.5% growth rate continues.

(B) What is the estimated population (to the nearest tenth of a billion) of India. in the year 2030?

Growth model : $\frac{dQ}{dt} = rQ$ ① $r = \frac{1.5}{100} = 0.015$

(แก้สมการแล้ว) $\Downarrow$ ② $Q(0) = 1.2$ billion

(A) $Q(t) = Q_0 e^{rt} = 1.2e^{0.015t}$

(B) หา $Q(20)$ (ตามปี 2010 set $t=0$
$\therefore$ 2030 set $t=20$)

$\therefore Q(20) = 1.2e^{0.015(20)}$

$= 1.2e^{0.3}$ พันล้านคน #

We now turn to another type of exponential growth: *radioactive decay* . In 1946,Willard Libby (who later received a Nobel Prize in chemistry) found that as long as a plant or animal is alive, radioactive carbon-14 is maintained at a constant level in its tissues. Once the plant or animal is dead, however, the radioactive carbon-14 diminishes by radioactive decay at a rate proportional to the amount present.

$$\frac{dQ}{dt} = rQ \qquad Q(0) = Q_0$$

This is another example of the exponential growth law. The continuous compound rate of decay for radioactive carbon-14 is 0.0001238, so $\boxed{r = -0.0001238,}$ since decay implies a negative continuous compound growth rate.

Example 9.20 A human bone fragment was found at an archaeological site in Africa. If 10% of the original amount of radioactive carbon-14 was present, estimate the age of the bone.

$\frac{dQ}{dt} = rQ$; ให้ $Q(t)$ แทนปริมาณของ Carbon-14 ณ เวลา t

Q_0 คือ ปริมาณ Carbon-14 เริ่มต้น $(Q(0)=Q_0)$

เมื่อเวลาผ่านไป t ปริมาณ Carbon-14 จะลดลงเหลือ 10% ของปริมาณเริ่มต้น $Q(t) = 0.1Q_0$

หา $\boxed{t = ?}$ ที่ $Q(t) = 0.1Q$

$\boxed{Q(t) = Q_0e^{rt}}$ $\overset{\text{Solve } t}{\Longleftrightarrow}$ $0.1\cancel{Q_0} = \cancel{Q_0}e^{rt}$

$$0.1 = e^{-0.0001238t}$$

Take ln; $\ln(0.1) = \ln(e^{-0.0001238t})$

$$-\ln 10 = -0.0001238t \Longleftrightarrow \boxed{t = \frac{1}{0.0001238}\ln 10}$$

Example 9.21 Half-life is the time required for a radioactive element to reduce its quantity by half.

Denote by T the half-life of a radioactive element. Use $y = y_0e^{-kt}$ to write T in terms of the decay constant k. If the half-life of radium-226 is 1600 years, find its decay constant.

ให้ Q_0 เป็นปริมาณสารเริ่มต้น หา $\boxed{r = ?}$

$\frac{dQ}{dt} = rQ$; ครึ่งชีวิต: $Q(1600) = \frac{1}{2}Q_0$

$\therefore \boxed{Q(t) = Q_0e^{rt}} \Longleftrightarrow \frac{1}{2}\cancel{Q_0} = Q(1600) = \cancel{Q_0}e^{1600r}$

$$\therefore \frac{1}{2} = e^{1600r}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{1600r})$$

$$-\ln 2 = 1600r \Longleftrightarrow \boxed{r = -\frac{\ln 2}{1600}} < 0 \ \#$$

9.6 Comparison of Exponential Growth Phenomena

The graphs and equations given in Figure below compare several widely used growth models. These models are divided into two groups: unlimited growth and limited growth. Following each equation and graph is a short (and necessarily incomplete) list of areas in which the models are used.

Table 1 Exponential Growth

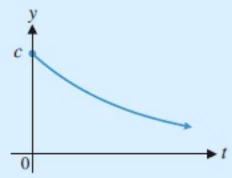
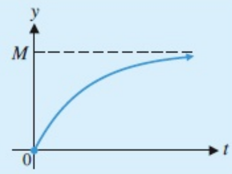
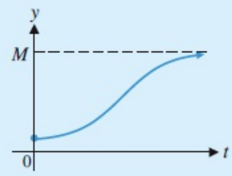
Description	Model	Solution	Graph	Uses
Unlimited growth: Rate of growth is proportional to the amount present	$\frac{dy}{dt} = ky$ $k, t > 0$ $y(0) = c$	$y = ce^{kt}$		• Short-term population growth (people, bacteria, etc.) • Growth of money at continuous compound interest • Price–supply curves
Exponential decay: Rate of growth is proportional to the amount present	$\frac{dy}{dt} = -ky$ $k, t > 0$ $y(0) = c$	$y = ce^{-kt}$		• Depletion of natural resources • Radioactive decay • Absorption of light in water • Price–demand curves • Atmospheric pressure (t is altitude)
Limited growth: Rate of growth is proportional to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = k(M - y)$ $k, t > 0$ $y(0) = 0$	$y = M(1 - e^{-kt})$		• Sales fads (for example, skateboards) • Depreciation of equipment • Company growth • Learning
Logistic growth: Rate of growth is proportional to the amount present and to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = ky(M - y)$ $k, t > 0$ $y(0) = \frac{M}{1 + c}$	$y = \frac{M}{1 + ce^{-kMt}}$		• Long-term population growth • Epidemics • Sales of new products • Spread of a rumor • Company growth

Figure 9.2: Exponential growth

9.6 Comparison of Exponential Growth Phenomena

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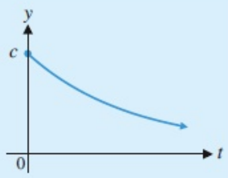
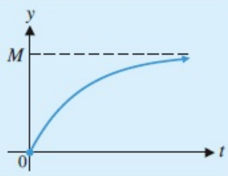
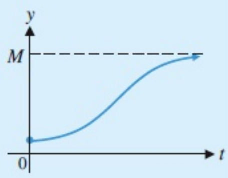
Exponential decay: Rate of growth is proportional to the amount present	$\frac{dy}{dt} = -ky$ $k, t > 0$ $y(0) = c$	$y = ce^{-kt}$		• Depletion of natural resources • Radioactive decay • Absorption of light in water • Price–demand curves • Atmospheric pressure (t is altitude)
Limited growth: Rate of growth is proportional to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = k(M - y)$ $k, t > 0$ $y(0) = 0$	$y = M(1 - e^{-kt})$		• Sales fads (for example, skateboards) • Depreciation of equipment • Company growth • Learning
Logistic growth: Rate of growth is proportional to the amount present and to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = ky(M - y)$ $k, t > 0$ $y(0) = \frac{M}{1 + c}$	$y = \frac{M}{1 + ce^{-kMt}}$		• Long-term population growth • Epidemics • Sales of new products • Spread of a rumor • Company growth

Figure 9.2: Exponential growth

CH5 Antiderivative (Integral) → สูตรพื้นฐาน
→ แทน u-substitution

CH6 Techniques of integration

→ by parts u, dv ; LIATE

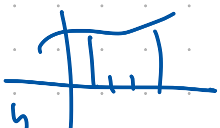
→ trigono ยกกำลัง $\int \sin^m x \cos^n x \, dx$

→ แทนค่าด้วยตรีโกณ $x = \sin\theta$, $x = \tan\theta$, $x = \sec\theta$

→ partial fraction +

CH7 definite integral

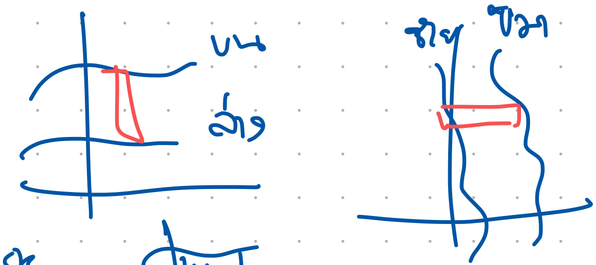
- Riemann sum ,



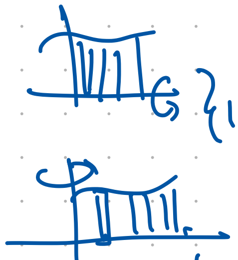
- Net-signed area ⇔ $\int_a^b f(x)\,dx$
- FTC
  - $\int_a^b f(x)\,dx = F(b) - F(a)$
  - $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- แทน u-substitution กับ definite integral

CH8 Geometry Application

- พ.ท.



- ปริมาตร → disk
  - → washer
  - → shell



{ เมื่อไหร่ใช้

- Improper integral → 3 type
  - $\int_{-\infty}^{0}$
  - $\int f(x)$
  - ① $+\varepsilon$

CH9 DEs → Check การเป็นคำตอบ

→ Separable

→ Linear Eq^{n}

→ Application