ทบทวน

(1) $y'' + 4y' + 8y = 0$

$r^2 + 4r + 8 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4(8)}}{2}$

$r = \dfrac{-4 \pm 4i}{2}$

$r = -2 \pm 2i$

$y(x) = e^{-2x}\left[C_1 \cos 2x + C_2 \sin 2x\right]$ #

(2) $9y'' + 6y' + y = 0$

$9r^2 + 6r + 1 = 0$

$(3r+1)(3r+1) = 0$

$r = -\frac{1}{3}$

$y(x) = C_1 e^{-\frac{1}{3}x} + C_2 x e^{-\frac{1}{3}x}$ #

$ay'' + by' + cy = 0 \longleftrightarrow ar^2 + br + c = 0$

$y_1(x) = e^{r_1 x}$

$y_2(x) = e^{r_2 x}$

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

$r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

CASE 1

$b^2 - 4ac > 0$

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

CASE 2

$b^2 - 4ac = 0$

$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$

CASE 3

$b^2 - 4ac < 0$

$r = \alpha \pm i\beta$

$y(x) = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$

Case III: $b^2 - 4ac < 0$

> If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is
>
> $$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

Initial-Value and Boundary-Value Problems

Initial value problem
ปัญหาค่าเริ่มต้น

An **initial-value problem** for the second-order Equation (1) or (2) consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad \boxed{y(0) = 1 \qquad y'(0) = 0}$$

$$r^2 + r - 6 = 0$$
$$(r+3)(r-2) = 0$$
$$r = 2, -3$$
$$\boxed{\therefore y(x) = C_1 e^{2x} + C_2 e^{-3x}}$$

$$1 = y(0) = C_1 + C_2 \iff C_1 + C_2 = 1$$
$$y'(x) = 2C_1 e^{2x} - 3C_2 e^{-3x}$$
$$\therefore 0 = y'(0) = 2C_1 - 3C_2 \iff 2C_1 = 3C_2 \iff C_1 = \frac{3}{2}C_2$$

$$\frac{3}{2}C_2 + C_2 = 1$$
$$\frac{5}{2}C_2 = 1$$
$$\boxed{C_2 = \frac{2}{5}}$$
$$\therefore C_1 = \left(\frac{3}{2}\right)\left(\frac{2}{5}\right) = \frac{3}{5}$$
$$\therefore \boxed{y(x) = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}}$$

EXAMPLE 6 Solve the initial-value problem

$$y'' + y' = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$r = 0, -1$$

$$\boxed{y(x) = c_1 + c_2 e^{-x}}$$

$$2 = y(0) = c_1 + c_2 \iff c_1 + c_2 = 2 \Rightarrow c_1 = 5$$

$$y'(x) = -c_2 e^{-x}$$

$$3 = y'(0) = -c_2 \iff -c_2 = 3$$

$$\therefore c_2 = -3$$

$$\therefore \boxed{y(x) = 5 - 3e^{-x}}$$

ฝึกบทที่สอง

A **boundary-value problem** for Equation (1) consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

$$r^2 + 2r + 1 = 0$$
$$(r+1)(r+1) = 0$$
$$r = -1$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 x e^{-x}}$$

$$1 = y(0) = c_1 \iff c_1 = 1$$

$$3 = y(1) = c_1 e^{-1} + c_2 e^{-1} \iff 3 = \frac{1}{e} + \frac{c_2}{e}$$

$$3e - 1 = c_2$$

$$\therefore y(x) = e^{-x} + (3e-1)xe^{-x}$$

Summary: Solutions of $ay'' + by' + cy = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y(x) = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$$ay'' + by' + cy = \boxed{G(x)} \longrightarrow$$ Case $G(x) = 0$ homogenous eqⁿ

$\boxed{G(x) \neq 0}$

$\boxed{1}$ $ay'' + by' + cy = 0 \leftarrow$ y ที่สอดคล้องกับ

เรียกว่า คำตอบประกอบ complementary equation

$y_c(x)$

$\boxed{2}$ หาคำตอบที่สอดคล้องกับ $G(x)$

↳ คำตอบที่เกิดจากการสมมติฟอร์มของ $G(x)$

$y_p(x)$

particular solution คำตอบเฉพาะ

Thm คำตอบทั่วไปของสมการเชิงอนุพันธ์ไม่เอกพันธ์ $ay'' + by' + cy = G(x)$ คือ

$$y(x) = y_c(x) + y_p(x)$$

คำตอบจาก $ay'' + by' + cy = 0$

คำตอบจากการสมมติform ที่สอดคล้องกับ $ay'' + by' + cy = G(x)$

Note $y_c(x), y_p(x)$ จะต้อง L.I. ตามหลักการซ้อนทับของคำตอบ "superposition principle"

Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \qquad (1)$$

where $a, b,$ and c are constants and G is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \qquad (2)$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation (1) and y_c is the general solution of the complementary Equation (2).

The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

EXAMPLE 1 Solve the equation $y'' + y' - 2y = x^2$.

Ex 1 $\quad y'' + y' - 2y = x^2$ → Form 1 $G(x)$ เป็นพหุนาม

เช่น $G(x) = x^2$

สมมติ $y_p(x) = Ax^2 + Bx + C$

$G(x) = 2x$

$y_p(x) = Ax + B$

$G(x) = 5x^3$

$y_p(x) = Ax^3 + Bx^2 + Cx + D$

$y_p(x)$ สอดคล้องกับ $y'' + y' - 2y = x^2$

① หา $y_c(x)$: คำตอบของ $y'' + y' - 2y = 0$

$$r^2 + r - 2 = 0$$
$$(r+2)(r-1) = 0$$
$$\therefore r = -2, 1$$

$$y_c(x) = c_1 e^{-2x} + c_2 e^{x}$$

② หา $y_p(x)$: $G(x) = x^2$

$$\left.\begin{aligned} \therefore y_p(x) &= Ax^2 + Bx + C \\ y_p'(x) &= 2Ax + B \\ y_p''(x) &= 2A \end{aligned}\right\} \text{แทนในโจทย์}$$

$$\therefore y'' + y' - 2y = (2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

เทียบ สปส :

$$-2A = 1 \iff 2A = -1, \; A = -\tfrac{1}{2}$$
$$2A - 2B = 0 \qquad B = -\tfrac{1}{2}$$
$$2A + B - 2C = 0$$
$$\hookrightarrow -1 - \tfrac{1}{2} = 2C \iff C = -\tfrac{3}{4}$$

$$\therefore y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

$$\therefore y(x) = y_c(x) + y_p(x)$$
$$= c_1 e^{-2x} + c_2 e^{x} + \left(-\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}\right)$$

Form 2 $G(x)$ เป็นฟังก์ชัน exponential

$G(x)=e^{3x}$, $y_p(x)=Ae^{3x}$ | $G(x)=e^{-5x}$, $y_p(x)=Ae^{-5x}$ | $G(x)=e^{\sqrt{2}x}$, $y_p(x)=Ae^{\sqrt{2}x}$

If $G(x)$ (the right side of Equation (1)) is of the form Ce^{kx}, where C and k are constants, then we take as a trial solution a function of the same form, $y_p(x)=Ae^{kx}$, because the derivatives of e^{kx} are constant multiples of e^{kx}.

EXAMPLE 2 Solve $y''+4y=e^{3x}$.

(1) หา $y_c(x)$: $y''+4y=0$

$r^2+4=0$

$r=0\pm 2i$

$y_c(x)=e^{0x}[c_1\cos 2x+c_2\sin 2x]$

$\therefore y_c(x)=c_1\cos 2x+c_2\sin 2x$

(2) หา $y_p(x)$: สมมติ $y_p(x)=Ae^{3x}$

$y_p'(x)=3Ae^{3x}$

$y_p''(x)=9Ae^{3x}$

$\therefore y''+4y=9Ae^{3x}+4Ae^{3x}=e^{3x}$

$13Ae^{3x}=e^{3x}$

เทียบสปส. $13A=1$

$A=\frac{1}{13}$

$y_p(x)=\frac{1}{13}e^{3x}$

$\therefore y(x)=y_c(x)+y_p(x)$

$=c_1\cos 2x+c_2\sin 2x+\frac{1}{13}e^{3x}$

CASE 3 $G(x)$ เป็น trigono.

$G(x)=\sin x$

$y_p(x)=A\cos x+B\sin x$

$G(x)=\cos 2x$

$y_p(x)=A\cos 2x+B\sin 2x$

If $G(x)$ is either $C\cos kx$ or $C\sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x)=A\cos kx+B\sin kx$$

$G(x)=\cos 2x+\sin x$

$y_p(x)=A\cos 2x+B\sin 2x+C\cos x+D\sin x$

EXAMPLE 3 Solve $y''+y'-2y=\sin x$.

(1) หา $y_c(x)$: $y''+y'-2y=0$

$r^2+r-2=0$

$(r+2)(r-1)=0$

$r=-2,1$

$\therefore y_c(x)=c_1e^{-2x}+c_2e^{x}$

(2) หา $y_p(x)$: สมมติ $y_p(x)=A\cos x+B\sin x$

$y_p'(x)=-A\sin x+B\cos x$

$y_p''(x)=-A\cos x-B\sin x$

$\therefore y''+y'-2y=(-A\cos x-B\sin x)+(-A\sin x+B\cos x)-2(A\cos x+B\sin x)=\sin x$

$(-3A+B)\cos x+(-A-3B)\sin x=\sin x$

เทียบสปส.: $\left.\begin{aligned}-3A+B&=0\\-A-3B&=1\end{aligned}\right\}\Rightarrow B=3A$

$-A-3(3A)=1 \iff A=-\frac{1}{10}$ ได้ $B=-\frac{3}{10}$

$\therefore y_p(x)=-\frac{1}{10}\cos x-\frac{3}{10}\sin x$

$\therefore y(x)=c_1e^{-2x}+c_2e^{x}+\left(-\frac{1}{10}\cos x-\frac{3}{10}\sin x\right)$

G(x) เป็นผลคูณ $\quad G(x) = x\cos 3x$

$Y_p(x) = (Ax+B)[C\cos 3x + D\sin 3x]$

$= (Ax+B)\cos 3x + (Cx+D)\sin 3x$

If $G(x)$ is **a product of functions of the preceding types**, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax+B)\cos 3x + (Cx+D)\sin 3x$$

If $G(x)$ is a **sum of functions of these types**, we use the easily verified principle of superposition, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad\qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

กำหนด $ay''+by'+cy = G(x)$ จงสมมติ $y_p(x)$ [สมมติก่อนว่า $y_c(x)$ กับ $y_p(x)$ L.I.]

① $y''+3y'+2y = x^2$

$$y_p(x) = Ax^2+Bx+C$$

② $y''-4y = xe^{x}+\cos 2x$

$$y_p(x) = (Ax+B)[Ce^{x}] + [D\cos 2x + E\sin 2x]$$

$$= (A_1x+B_1)e^{x} + [D\cos 2x + E\sin 2x]$$

$$(A_1 = AC,\ B_1 = BC)$$

③ $y''+9y' = xe^{-x}\cos \pi x$

$$y_p(x) = (Ax+B)(Ce^{-x})[D\cos \pi x + E\sin \pi x]$$

$$= (A_1x+B_1)e^{-x}[D\cos \pi x + E\sin \pi x]$$

EXAMPLE 5 Solve $y'' + y = \sin x$.

We summarize the method of undetermined coefficients as follows:

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation.)

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.