ทบทวน

$$ay'' + by' + cy = G(x)$$

- $ay'' + by' + cy = 0$

พิจารณาสมการช่วย $ar^2 + br + c = 0$

- $b^2 - 4ac > 0$; $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- $b^2 - 4ac = 0$; $y(x) = c_1 e^{rx} + c_2 x e^{rx}$
- $b^2 - 4ac < 0$; $y(x) = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$
  $(r = \alpha \pm i\beta)$

- $ay'' + by' + cy = G(x)$

  1. $y_c(x)$: คำตอบที่สอดคล้องกับ $ay'' + by' + cy = 0$
  2. $y_p(x)$ ขึ้นอยู่กับการสมมติ

     - พหุนามดีกรี n: $G(x) = \underbrace{x^3 + x^2 + 1}$; $y_p(x) = Ax^3 + Bx^2 + Cx + D$
     - exponential: $G(x) = 5e^{-4x}$; $y_p(x) = Ae^{-4x}$
       $G(x) = e^{-4}$; $y_p(x) = A$
     - trigonometric fn: $G(x) = 5\cos 2x$; $y_p(x) = A\cos 2x + B\sin 2x$
       $G(x) = 3\sin x + \cos x$; $y_p(x) = A\cos x + B\sin x$
       $G(x) = 3\cos 2x + 4\sin x$;
       $y_p(x) = [A\cos 2x + B\sin 2x] + [C\cos x + D\sin x]$
     - ผลบวกของฟังก์ชัน: $G(x) = \underline{\cos x} + e^{-x}$
       $y_p(x) = \underline{[A\cos x + B\sin x]} + Ce^{-x}$
     - ผลคูณของฟังก์ชัน: $G(x) = \underline{\underline{x}}\cos x$
       $y_p(x) = (Ax + B)[C\cos x + D\sin x]$ ✓ — (1)
       $= (Ax + B)\cos x + (Cx + D)\sin x$ — (2)
       (Solve ง่ายกว่า)

If $G(x)$ is **a product of functions of the preceding types**, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax + B)\cos 3x + (Cx + D)\sin 3x$$

If $G(x)$ is a **sum of functions of these types**, we use the easily verified principle of superposition, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad\qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

(1) หา $y_c(x)$: $r^2 - 4 = 0$

$\therefore r = \pm 2$

$$\boxed{y_c(x) = c_1 e^{2x} + c_2 e^{-2x}}$$

(2) หา $y_p(x)$: สมมติ $y_p(x) = (Ax+B)e^x + [C\cos 2x + D\sin 2x]$

ข้อสังเกตจาก $y'' - 4y = G(x)$; $y_p'(x) = (Ax+B)e^x + Ae^x + [-2C\sin 2x + 2D\cos 2x]$

$y_p''(x) = (Ax+B)e^x + Ae^x + Ae^x + [-4C\cos 2x - 4D\sin 2x]$

แทน: $y'' - 4y = [(Ax+B)e^x + 2Ae^x - 4C\cos 2x - 4D\sin 2x] - 4(Ax+B)e^x - 4C\cos 2x - 4D\sin 2x$

$= xe^x + \cos 2x$

$\therefore (A - 4A)xe^x + [B + 2A - 4B]e^x + (-4C - 4C)\cos 2x + (-4D - 4D)\sin 2x = xe^x + \cos 2x$

เทียบ สปส:

$$\left.\begin{aligned} -3A &= 1 \\ 2A - 3B &= 0 \\ -8C &= 1 \\ -8D &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= -\tfrac{1}{3} \\ B &= -\tfrac{2}{9} \\ C &= -\tfrac{1}{8} \\ D &= 0 \end{aligned}$$

$$\therefore \boxed{y_p(x) = \left[-\tfrac{1}{3}x - \tfrac{2}{9}\right]e^x - \tfrac{1}{8}\cos 2x}$$

$\therefore y(x) = y_c(x) + y_p(x)$

$$y(x) = \boxed{c_1 e^{2x} + c_2 e^{-2x}} + \boxed{\left[-\tfrac{1}{3}x - \tfrac{2}{9}\right]e^x - \tfrac{1}{8}\cos 2x}$$

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

EXAMPLE 5 Solve $y'' + y = \sin x$.

① $y_c(x)$: $r^2 + 1 = 0$

$r = \pm i$

$\therefore \boxed{y_c(x) = C_1 \cos x + C_2 \sin x}$

② $y_p(x) = x[A\cos x + B\sin x] = Ax\cos x + Bx\sin x$

$\therefore y_p'(x) = -Ax\sin x + A\cos x + Bx\cos x + B\sin x$

$y_p''(x) = -Ax\cos x - A\sin x - A\sin x - Bx\sin x + B\cos x + B\cos x$

$= -Ax\cos x - 2A\sin x - Bx\sin x + 2B\cos x$

$\therefore y_p'' + y_p = \sin x \iff [\cancel{-Ax\cos x} - 2A\sin x \cancel{- Bx\sin x} + 2B\cos x] + \cancel{Ax\cos x} + \cancel{Bx\sin x} = \sin x$

$-2A\sin x + 2B\cos x = \sin x$

$\left.\begin{array}{l} -2A = 1 \\ 2B = 0 \end{array}\right\} A = -\frac{1}{2},\ B = 0$

$\boxed{y_p(x) = -\frac{1}{2}x\cos x}$

$\therefore \boxed{y(x) = \boxed{C_1\cos x + C_2\sin x} \boxed{-\frac{1}{2}x\cos x}}$

We summarize the method of undetermined coefficients as follows:

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation.)

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.

① $y_c(x)$: $r^2 - 4r + 13 = 0$

$r = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$

$y_c(x) = e^{2x}[C_1\cos 3x + C_2\sin 3x]$

② $y_p(x) = x e^{2x}[A\cos 3x + B\sin 3x]$

แบบฝึกหัดทบทวน เรื่อง สมการเชิงอนุพันธ์อันดับสองไม่เอกพันธุ์

ชื่อ-สกุล.. รหัสนักศึกษา............................ ลำดับที่...........

สรุปการสมมติ $y_p(x)$ ตามชนิดของ $G(x)$

ฟังก์ชัน $G(x)$		การสมมติ $y_p(x)$
ฟังก์ชันเดี่ยว	ฟังก์ชันพหุนามดีกรี n $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$	$A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$
	ฟังก์ชันเอกซ์โปเนนเชียล Ae^{kx}	Be^{kx}
	ฟังก์ชันตรีโกณมิติ $A\sin mx$ หรือ $A\cos mx$ หรือ $A\sin mx + B\cos mx$	$A_1\sin mx + B_1\cos mx$
ผลคูณของฟังก์ชัน	$P_n(x)e^{kx}$	$(A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0)e^{kx}$
	$P_n(x)\sin mx$ หรือ $P_n(x)\cos mx$ หรือ $P_n(x)\sin mx + P_n(x)\cos mx$	$(A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0)\sin mx$ $+(B_nx^n + B_{n-1}x^{n-1} + \ldots + B_1x + B_0)\cos mx$
	$ae^{kx}\sin mx$, $be^{kx}\cos mx$ หรือ $e^{kx}[a\sin mx + b\cos mx]$	$e^{kx}[A\sin mx + B\cos mx]$

1. จงกำหนด $y_p(x)$ ที่สอดคล้องกับ $G(x)$ ที่กำหนดให้ต่อไปนี้

(a) $G(x) = 2x^3 - x + 3$ $\qquad y_p(x) =$ $Ax^3 + Bx^2 + Cx + D$

(b) $G(x) = 2e^{-x}$ $\qquad y_p(x) =$ Ae^{-x}

(c) $G(x) = \sin 2x$ $\qquad y_p(x) =$ $A\sin 2x + B\cos 2x$

(d) $G(x) = 5\cos 4x$ $\qquad y_p(x) =$ $A\cos 4x + B\sin 4x$

(e) $G(x) = \sin 2x + \cos 2x$ $\qquad y_p(x) =$ $A\cos 2x + B\sin 2x$

(f) $G(x) = \sin 2x + \cos 3x$ $\qquad y_p(x) =$ $[A\cos 2x + B\sin 2x] + [C\cos 3x + D\sin 3x]$

(g) $G(x) = x\sin 2x$ $\qquad y_p(x) =$ $(Ax+B)\cos 2x + (Cx+D)\sin 2x$

(h) $G(x) = x^2e^{-x}$ $\qquad y_p(x) =$ $(Ax^2 + Bx + C)e^{-x}$

(i) $G(x) = 3e^x + ex^2 + 1$ $\qquad y_p(x) =$ $Ae^x + (Bx^2 + Cx + D)$

(j) $G(x) = (2e^{3x} + 3)\sin x + xe^{-x}$ $\qquad y_p(x) =$ $e^{3x}[A\cos x + B\sin x] +$ $[C\cos x + D\sin x] + (Ex+F)e^{-x}$

$= 2e^{3x}\sin x + 3\sin x + xe^{-x}$

2. จงหา $y_c(x)$ และสมมติ $y_p(x)$ โดยวิธีเทียบสัมประสิทธิ์ (ไม่ต้องคำนวณค่าคงตัว)

(a) $16y'' - 8y' + y = xe^{x/4} + e^{4x}$

$16r^2 - 8r + 1 = 0$
$(4r-1)(4r-1) = 0$
$r = 1/4$

$y_c(x) =$ $C_1 e^{\frac{x}{4}} + C_2 x e^{\frac{x}{4}}$

$y_p(x) =$ $Ae^{4x} + x^2(Bx + C)e^{x/4}$

(b) $y'' + 8y = 7\cos 2\sqrt{2}x$

$r^2 + 8 = 0$
$r = \pm 2\sqrt{2}i$

$y_c(x) =$ $C_1 \cos 2\sqrt{2}x + C_2 \sin 2\sqrt{2}x$

$y_p(x) =$ $x[A\cos 2\sqrt{2}x + B\sin 2\sqrt{2}x]$

(c) $y'' - 2y' - 9y = x\sin 10x + 2\pi$

$r^2 - 2r - 9 = 0$
$r = \frac{2 \pm \sqrt{4 - 4(-9)}}{2} = \frac{2 \pm 2\sqrt{10}}{2} = 1 \pm \sqrt{10}$

$y_c(x) =$ $C_1 e^{(1+\sqrt{10})x} + C_2 e^{(1-\sqrt{10})x}$

$y_p(x) =$ $(Ax + B)\cos 10x + (Cx + D)\sin 10x + E$

(d) $y'' + 3y' = x + 1 + 2\sin x$

$r^2 + 3r = 0$
$r = 0, -3$

$y_c(x) =$ $C_1 + C_2 e^{-3x}$

$y_p(x) =$ $x(Ax + B) + [C\cos x + D\sin x]$

(e) $y'' + 4y = x\sin 2x + e^{2x}\cos 2x + 5$

$r^2 + 4 = 0$
$r = \pm 2i$

$y_c(x) =$ $C_1 \cos 2x + C_2 \sin 2x$

$y_p(x) =$ $x(Ax + B)\cos 2x + x(Cx + D)\sin 2x + e^{2x}[E\cos 2x + F\sin 2x] + G$

(f) $y'' + 3y' + 2y = x^2 + xe^{-x} + e^{-2x}\cos x$

$y_c(x) =$ ______

$y_p(x) =$ ______

(g) $y'' + 2y' = xe^{2x} - 10$

$y_c(x) =$ ______

$y_p(x) =$ ______

(h) $y'' - 12y' + 16y = \frac{3}{2}e^{6x} - e^{-6x}$

$y_c(x) =$ ______

$y_p(x) =$ ______

(i) $y'' - 8y' + 17y = x\sin x - 6e^{4x}$

$y_c(x) =$ ______

$y_p(x) =$ ______

(j) $y'' + 2y' + 2y = e^x \sin 3x$

$y_c(x) =$ __________

$y_p(x) =$ __________

$r^2 - 4r + 4 = 0 \;;\; r = 2$

(k) $y'' - 4y' + 4y = \underline{xe^{2x}} + \underset{\sim}{5}$

$y_c(x) =$ $c_1 e^{2x} + c_2 x e^{2x}$

$y_p(x) =$ $x^2(Ax+B)e^{2x} + C$

(l) $y'' + y = x \sin x$

$y_c(x) =$ __________

$y_p(x) =$ __________

(m) $y'' - 2y' - 3y = 7e^{3x} + x^2$

$y_c(x) =$ __________

$y_p(x) =$ __________

(n) $y'' - 2y' - 3y = 6e^{-3x} + 7e^{4x} + 5x$

$y_c(x) =$ __________

$y_p(x) =$ __________

(o) $y'' - 4y' + 4y = 10e^{2x} + \sin 7x$

$y_c(x) =$ __________

$y_p(x) =$ __________

(p) $y'' - 2y' + 5y = 8e^x \sin 2x$

$y_c(x) =$ __________

$y_p(x) =$ __________

3. จงหาคำตอบทั่วไปของสมการ $y'' + y = 10xe^x$