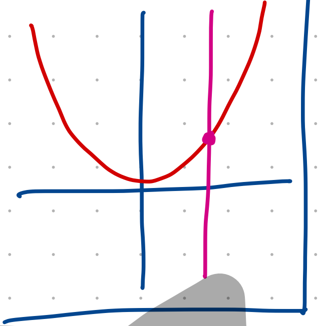
ฟังก์ชัน

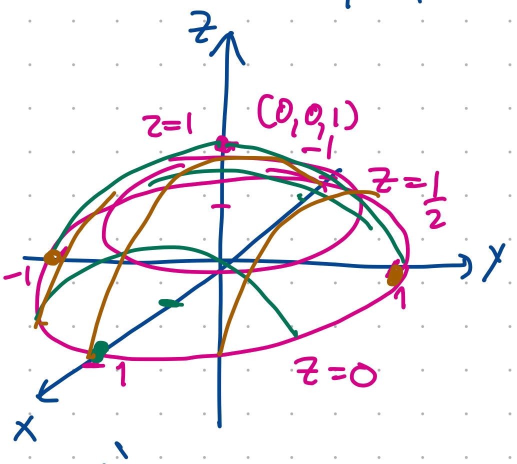


Trace of Surface

"mesh" line that results when a surface is cut by a plane.

Ex 4 (b) $f(x,y) = \sqrt{1-x^2-y^2}$

$z = \sqrt{1-x^2-y^2}$



Set $z=0$; $0 = \sqrt{1-x^2-y^2}$

$x^2+y^2 = 1$

$z=1$; $1 = \sqrt{1-x^2-y^2}$

$x^2+y^2 = 0$

$z = \frac{1}{2}$; $\frac{1}{2} = \sqrt{1-x^2-y^2}$

$\frac{1}{4} = 1-x^2-y^2$

$x^2+y^2 = 1-\frac{1}{4} = \frac{3}{4}$

ครึ่งทรงกลม

"hemisphere"

ลองตัดตามระนาบแนวอื่น : $x=0$ $z = \sqrt{1-y^2}$; $z \geq 0$ $x = -\frac{1}{2}$

$z^2 = 1-y^2$

$y^2+z^2 = 1$; $z \geq 0$

$x=1$ $z = \sqrt{\cancel{1}-\cancel{1}-y^2}$

$z^2 = -y^2$

$z^2+y^2 = 0$; $z \geq 0$

$x = \frac{1}{2}$ $z = \sqrt{1-\frac{1}{4}-y^2}$

$z^2 = \frac{3}{4} - y^2$

$y^2+z^2 = \frac{3}{4}$; $z \geq 0$

$y=0$ $z = \sqrt{1-x^2}$; $z \geq 0$

$z^2 = 1-x^2$

$x^2+z^2 = 1$; $z \geq 0$

$y=1$

$y = \frac{1}{2}$

Ex 4.c $f(x,y) = -\sqrt{x^2+y^2}$

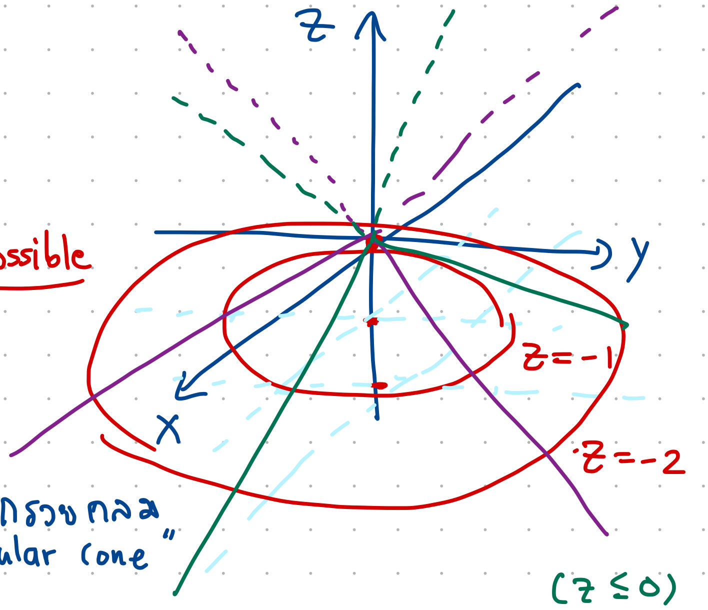
ตามระนาบ XY

$z=0$; $0 = -\sqrt{x^2+y^2}$
$x^2+y^2=0$

$z=1$; $1 = -\sqrt{x^2+y^2}$
$-1 = \sqrt{x^2+y^2}$ impossible

$z=-1$; $-1 = -\sqrt{x^2+y^2}$
$1 = \sqrt{x^2+y^2}$
$x^2+y^2 = 1$

$z=-2$; $-2 = -\sqrt{x^2+y^2}$
$x^2+y^2 = 4$



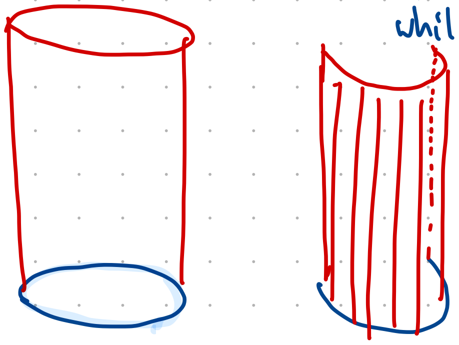
กรวยกลม "Circular cone"

ตามระนาบ YZ $x=0$; $z = -\sqrt{0+y^2}$
$z^2 = y^2$
$z^2 - y^2 = 0$
$(z-y)(z+y) = 0$
$\therefore z=y$ หรือ $z=-y$
$(z \le 0)$

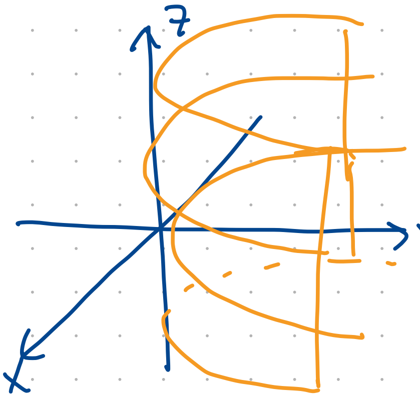
ตามระนาบ XZ $y=0$; $z = -\sqrt{x^2+0}$
$z^2 = x^2$
$(z-x)(z+x) = 0$
$\therefore z = x$ หรือ $z = -x$
$(z \le 0)$

Cylindrical surface

: A cylinder is a surface that is generated by moving a stright line along a given planar curve while holding the parallel line to a given fixed line.



Ex $y = x^2$ (ในสามมิติ)



ทรงกระบอกตามแกน z

ข้อสังเกต มีตัวแปรเพียงสองตัวแปรปรากฏในสมการนั้น ตัวแปรใดหายไป แสดงว่า ทรงกระบอกจะวิ่งตามแกนนั้น

2. Graphs of functions of two variables

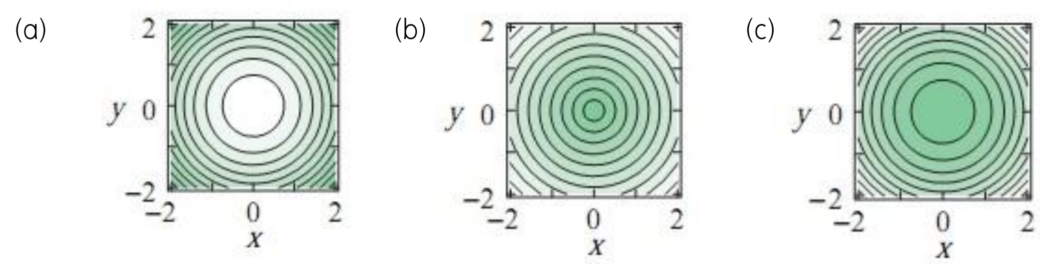
Recall that for a function f of one variable, the graph of $f(x)$ in the xy-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the **graph** of $f(x, y)$ in xyz-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

Example 4 In each part, describe the graph of the function in an xyz-coordinate system.

(a) $f(x,y) = 1 - x - \frac{1}{2}y$ (b) $f(x,y) = \sqrt{1 - x^2 - y^2}$ (c) $f(x,y) = -\sqrt{x^2 + y^2}$

3. Level curve

We are all familiar with the topographic (or contour) maps in which a three-dimensional landscape, such as a mountain range, is represented by two-dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 1. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 1, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.



20. Sketch the level curve $z=k$ for the specified values of k.

(a) $z=x^2+y^2;\ k=0,1,2,3,4$

(b) $z=y/x;\ k=-2,-1,0,1,2$

(c) $z=x^2+y;\ k=-2,-1,0,1,2$

(d) $z=x^2+9y^2;\ k=0,1,2,3,4$

(e) $z=x^2-y^2;\ k=-2,-1,0,1,2$

(f) $z=y\csc x;\ k=-2,-1,0,1,2$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 914-917)

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5. Quadric surfaces

201207 Solid Analytic Geometry

The equation in an xyz-coordinate system is

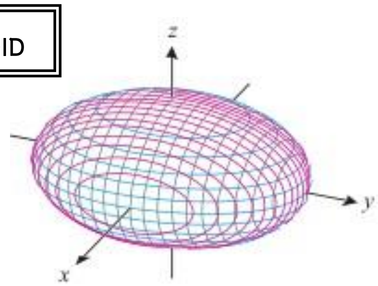
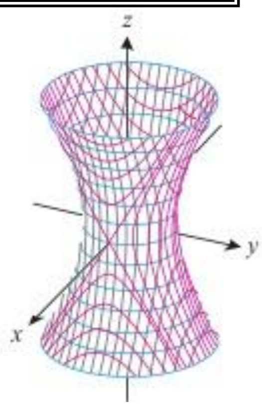
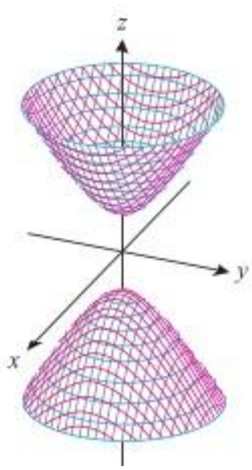
$$Ax^2+By^2+Cz^2+\boxed{Dxy+Exz+Fyz}+Gx+Hy+Iz+J=0$$

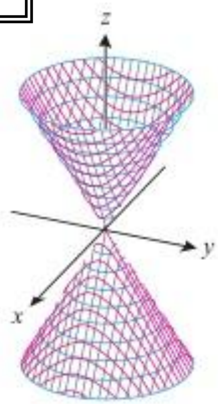
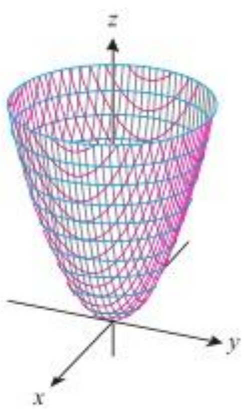
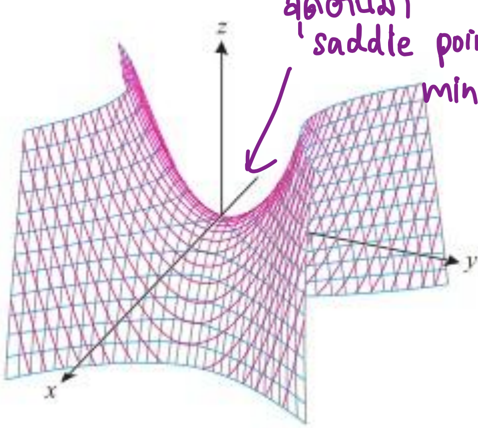
which is called a **second-degree equation in** x, y **and** z. The graphs of such equations are called **quadric surfaces** or sometimes **quadrics**.

Six common types of quadric surfaces are shown in Table 1—ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. (The constants a, b, and c that appear in the equations in the table are assumed to be positive.) Observe that none of the quadric surfaces in the table have cross-product terms in their equations. This is because of their orientations relative to the coordinate axes. Later in this section we will discuss other possible orientations that produce equations of the quadric surfaces with no cross-product terms. In the special case where the elliptic cross sections of an elliptic cone or an elliptic paraboloid are circles, the terms circular cone and circular paraboloid are used.

Table 1

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 823)

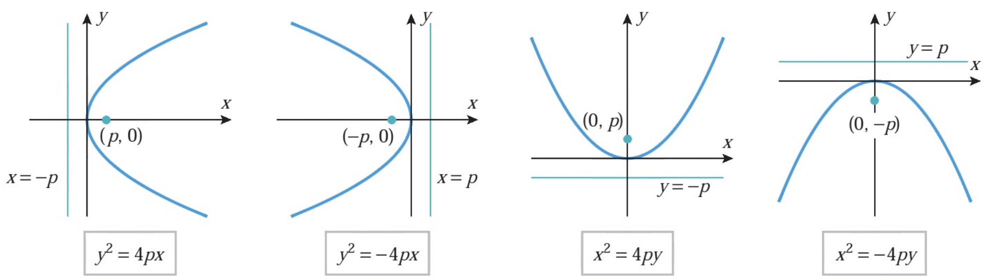
surface	equation
ELLIPSOID 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.
HYPERBOLOID OF ONE SHEET 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ The trace in the xy –plane is an ellipse, as are the traces in planes parallel to the xy –plane. The traces in the yz –plane and xz –plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x – or y –intercepts. At these intercepts the traces are pairs of intersecting lines.
HYPERBOLOID OF TWO SHEET 	$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ There is no trace in the xy –plane. In planes parallel to the xy –plane that intersect the surface in more than one point the traces are ellipses. In the yz – and xz –planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.

surface	equation
ELLIPTIC CONE 	$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$ The trace in the xy –plane is a point (the origin), and the traces in planes parallel to the xy –plane are ellipses. The traces in the yz –and xz –planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
ELLIPTIC PARABOLOID 	$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$ The trace in the xy –plane is a point (the origin), and the traces in planes parallel to and above the xy –plane are ellipses. The traces in the yz –and xz –planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLIC PARABOLOID จุดอานม้า saddle point / minimax point 	$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$ The trace in the xy –plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy –plane are hyperbolas. The hyperbolas above the xy –plane open in the y –direction, and those below in the x –direction. The traces in the yz –and xz –planes are parabolas, as are the traces in planes parallel to these.

ทบทวนภาคตัดกรวย

206112 QUICK REFERENCE

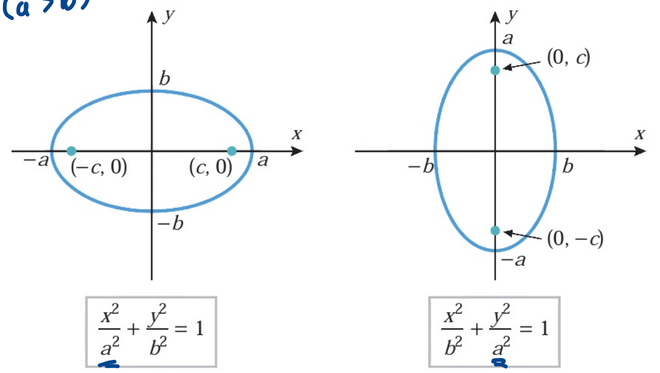
PARABOLAS IN STANDARD POSITION



ELLIPSES IN STANDARD POSITION

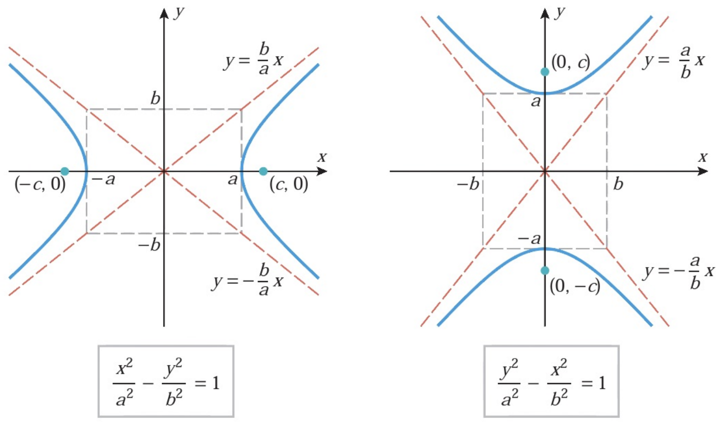
(a > b)

$c = \sqrt{a^2 - b^2}$



HYPERBOLAS IN STANDARD POSITION

เงื่อนไข: $c = \sqrt{a^2 + b^2}$



ภาคตัดกรวยแบบเลื่อนขนาน

Parabolas with vertex (h, k) and axis parallel to x-axis

$$(y-k)^2 = 4p(x-h) \quad \text{[Opens right]}$$
$$(y-k)^2 = -4p(x-h) \quad \text{[Opens left]}$$

Parabolas with vertex (h, k) and axis parallel to y-axis

$$(x-h)^2 = 4p(y-k) \quad \text{[Opens up]}$$
$$(x-h)^2 = -4p(y-k) \quad \text{[Opens down]}$$

Ellipse with center (h, k) and major axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad [b < a]$$

Ellipse with center (h, k) and major axis parallel to y-axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \quad [b < a]$$

Hyperbola with center (h, k) and focal axis parallel to x-axis

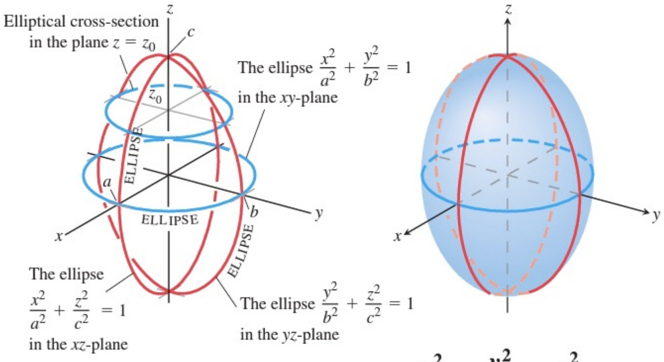
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Hyperbola with center (h, k) and focal axis parallel to y-axis

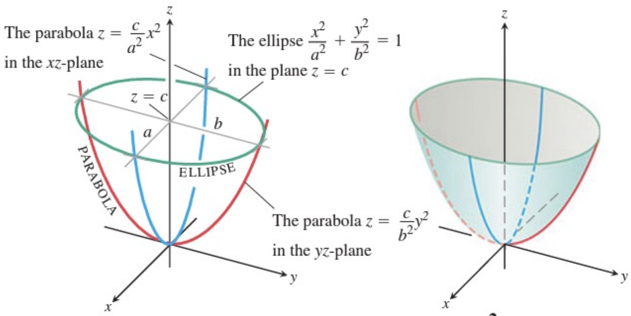
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

206112 QUICK REFERENCE

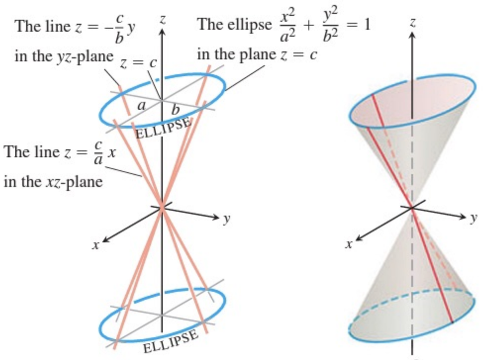
Quadric Surfaces



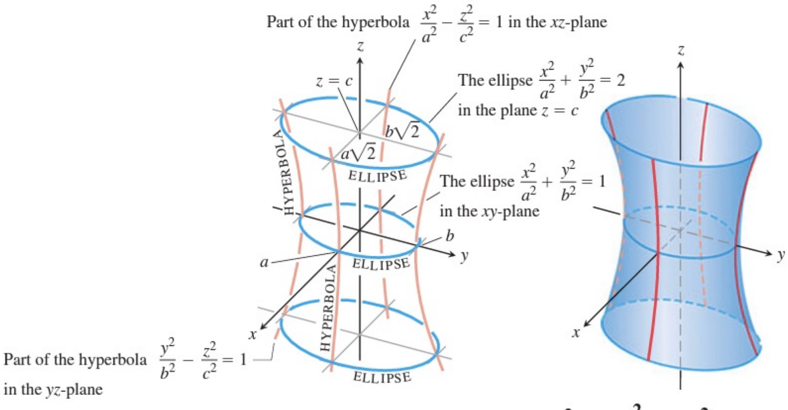
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



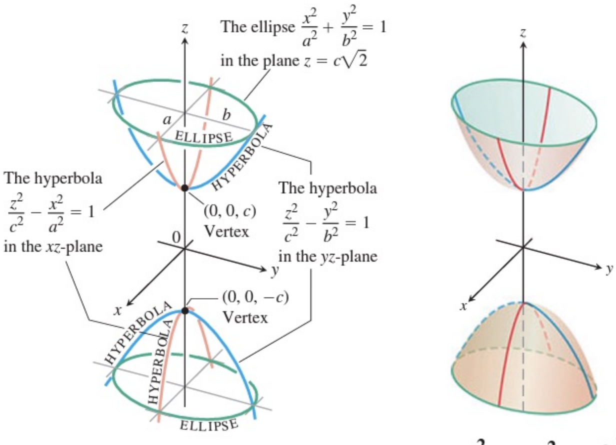
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



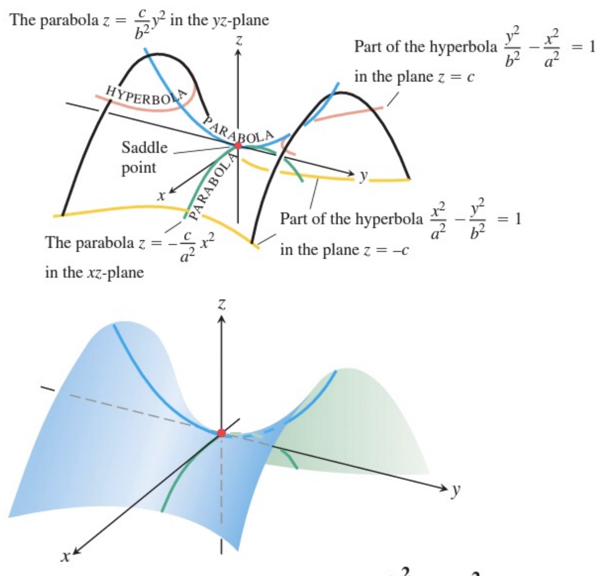
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

ที่มา : หนังสือ Thomas Calculus

5.1 Techniques for graphing quadric surfaces

Accurate graphs of quadric surfaces are best left for graphing utilities. However, the techniques that we will now discuss can be used to generate rough sketches of these surfaces that are useful for various purposes.

A rough sketch of an **ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the coordinate axes, and then sketching the elliptical traces in the coordinate planes. Example 7 illustrates this technique.

Example 7 Sketch the ellipsoid $\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$

หา x-intercept $(y,z=0)$ $\frac{x^2}{4}=1 \Leftrightarrow x=\pm 2$

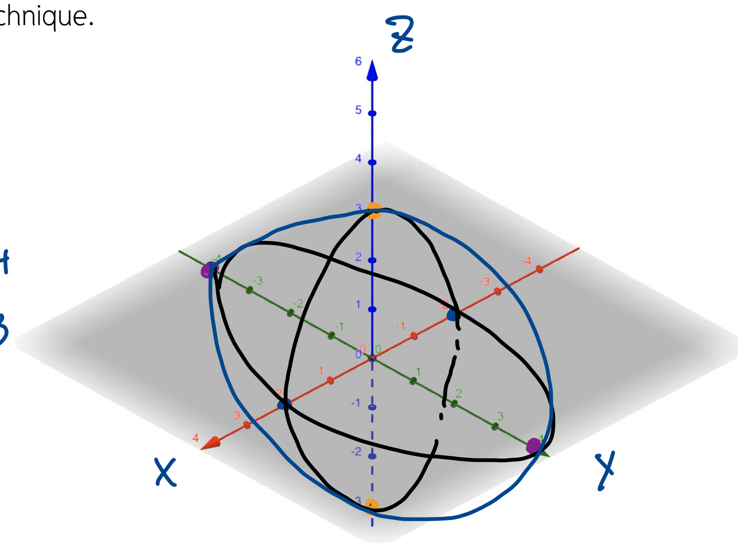
y-intercept $(x,z=0)$ $\frac{y^2}{16}=1 \Leftrightarrow y=\pm 4$

z-intercept $(x,y=0)$ $\frac{z^2}{9}=1 \Leftrightarrow z=\pm 3$

CHECK XY $(z=0)$ $\frac{x^2}{4}+\frac{y^2}{16}=1$

YZ $(x=0)$ $\frac{y^2}{16}+\frac{z^2}{9}=1$

XZ $(y=0)$ $\frac{x^2}{4}+\frac{z^2}{9}=1$



A rough sketch of a **hyperboloid of one sheet**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first sketching the elliptical trace in the xy-plane, then the elliptical traces in the planes $z=\pm c$, and then the hyperbolic curves that join the endpoints of the axes of these ellipses. The next example illustrates this technique.

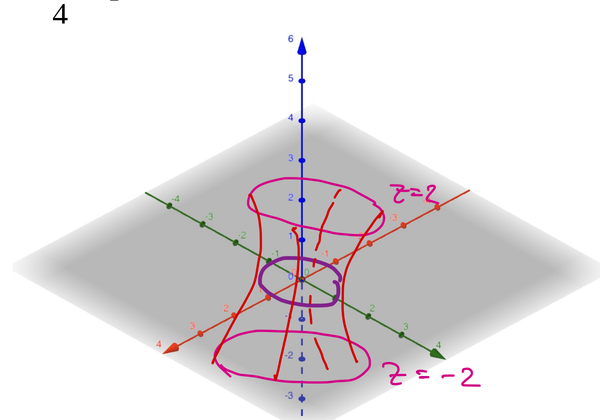
Example 8 Sketch the graph of the hyperboloid of one sheet $x^2+y^2-\frac{z^2}{4}=1$

ระนาบ XY $z=0$: $x^2+y^2=1$

ตามระนาบ $z=\pm c$; $z=\pm 2$

$z=2$: $x^2+y^2=2$

$z=-2$; $x^2+y^2=2$



A rough sketch of the **hyperboloid of two sheet**

$$\boxed{\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (a>0, b>0, c>0)}$$

can be obtained by first plotting the intersections with the z-axis, then sketching the elliptical traces in the planes $z=\pm 2c$, and then sketching the hyperbolic traces that connect the z-axis intersections and the endpoints of the axes of the ellipses. (It is not essential to use the planes $z=\pm 2c$, but these are good choices since they simplify the calculations slightly and have the right spacing for a good sketch.) The next example illustrates this technique.

Example 9 Sketch the graph of the hyperboloid of two sheet $z^2-x^2-\frac{y^2}{4}=1$

$z=0$ ไม่เกิดเส้นตัด ; $0-x^2-\frac{y^2}{4}=1$

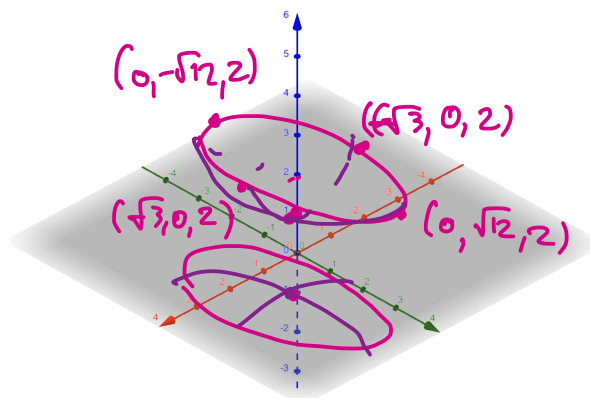
หา z = intercept ให้ $x, y=0 \iff z=\pm 1$

รอยตัด $\underline{z=\pm 2c}$

$z=2$; $4-x^2-\frac{y^2}{4}=1 \iff x^2+\frac{y^2}{4}=3$

$\frac{x^2}{3}+\frac{y^2}{12}=1$

$z=-2$; ได้เหมือนกัน



A rough sketch of the **elliptic cone**

$$\boxed{z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad (a>0, b>0)}$$

can be obtained by first sketching the elliptical traces in the planes $z=\pm 1$ and then sketching the linear traces that connect the endpoints of the axes of the ellipses. The next example illustrates this technique.

Example 10 Sketch the graph of the elliptic cone $z^2=x^2+\frac{y^2}{4}$

A rough sketch of the **elliptic paraboloid**

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (a > 0, b > 0)$$

can be obtained by first sketching the elliptical trace in the plane $z = 1$ and then sketching the parabolic traces in the vertical coordinate planes to connect the origin to the ends of the axes of the ellipse. The next example illustrates this technique.

Example 11 Sketch the graph of the elliptic paraboloid $z = \frac{x^2}{4} + \frac{y^2}{9}$

A rough sketch of the **hyperbolic paraboloid**

$$z^2 = \frac{y^2}{b^2} - \frac{x^2}{a^2} \qquad (a > 0, b > 0)$$

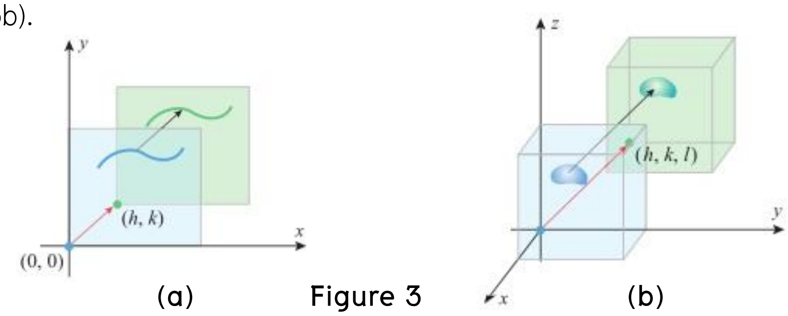
can be obtained by first sketching the two parabolic traces that pass through the origin (one in the plane $x = 0$ and the other in the plane $y = 0$). After the parabolic traces are drawn, sketch the hyperbolic traces in the planes $z = \pm 1$ and then fill in any missing edges. The next example illustrates this technique.

Example 12 Sketch the graph of the hyperbolic paraboloid $z = \frac{y^2}{4} - \frac{x^2}{9}$

5.2 Translations of quadric surfaces

A conic in an xy–coordinate system can be translated by substituting $x-h$ for x and $y-k$ for y in its equation. To understand why this works, think of the xy–axes as fixed and think of the plane as a transparent sheet of plastic on which all graphs are drawn. When the coordinates of points are modified by substituting $(x-h, y-k)$ for (x, y), the geometric effect is to translate the sheet of plastic (and hence all curves) so that the point on the plastic that was initially at $(0,0)$ is moved to the point (h,k) (see Figure 3a).

For the analog in three dimensions, think of the xyz–axes as fixed and think of 3–space as a transparent block of plastic in which all surfaces are embedded. When the coordinates of points are modified by substituting $(x-h, y-k, z-l)$ for (x, y, z) the geometric effect is to translate the block of plastic (and hence all surfaces) so that the point in the plastic block that was initially at $(0,0,0)$ is moved to the point (h,k,l) (see Figure 3b).



(a) **Figure 3** (b)

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 827)

Example 13 Describe the surface $z=(x-1)^2+(y+2)^2+3$

Example 14 Describe the surface $4x^2+4y^2+z^2+8y-4z=-4$

5.2 Techniques for identifying quadric surfaces

The equations of the quadric surfaces in Table 1 have certain characteristics that make it possible to identify quadric surfaces that are derived from these equations by reflections. These identifying characteristics, which are shown in Table 2, are based on writing the equation of the quadric surface so that all of the variable terms are on the left side of the equation and there is a 1 or a 0 on the right side. These characteristics do not change when the surface is reflected about a coordinate plane or planes of the form $x = y, x = z,$ or $y = z,$ thereby making it possible to identify the reflected quadric surface from the form of its equation.

Table 2

identifying a quadric surface from the form of its equation

equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
characteristic	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
classification	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

Example 15 Identify the surfaces

(a) $3x^2 - 4y^2 + 12z^2 + 12 = 0$ (b) $4x^2 - 4y + z^2 = 0$