$w = f(x, y, z)$

↑ ตัวแปรตาม

กราฟ (4 มิติ)

⇓

level surface (3 มิติ)

4. Level surfaces

Observe that the graph of $y = f(x)$ is a curve in 2–space, and the graph of $z = f(x, y)$ is a surface in 3–space, so the number of dimensions required for these graphs is one greater than the number of independent variables. Accordingly, there is no "direct" way to graph a function of three variables since four dimensions are required. However, if k is a constant, then the graph of the equation $f(x, y, z) = k$ will generally be a surface in 3–space (e.g., $x^2 + y^2 + z^2 = 1$ the graph of is a sphere), which we call the **level surface with constant** k. Some geometric insight into the behavior of the function f can sometimes be obtained by graphing these level surfaces for various values of k.

$f(x, y, z) = k$

สมการของ level surface

Example 6 Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

level surface : $f(x,y,z) = k$

$k < 0$; $x^2 + y^2 + z^2 = k$ ไม่เกิดกราฟ

$k = 0$; $x^2 + y^2 + z^2 = 0$ จุด $(0,0,0)$

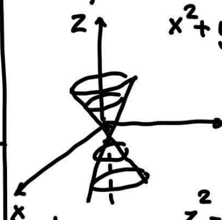
$k > 0$; $x^2 + y^2 + z^2 = k$ ทรงกลม รัศมี $\sqrt{k}$ จุดศ.ก $(0,0,0)$

level surface:

$f(x,y,z) = k$

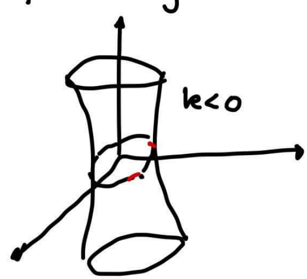
$k = 0$; $z^2 - x^2 - y^2 = 0$

$x^2 + y^2 = z^2 \Rightarrow$ กรวยกลม



$k > 0$; $z^2 - x^2 - y^2 = k$

$k < 0$; $z^2 - x^2 - y^2 = k$

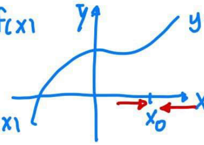


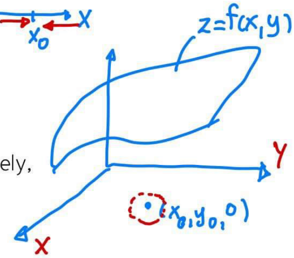
1 ตัวแปร : $y = f(x)$, $\lim_{x \to x_0} f(x)$

$\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x)$

2 ตัวแปร : $z = f(x,y)$

$\lim_{(x,y) \to (x_0,y_0)} f(x,y)$





6. Limits and continuity

6.1 Limits along curves

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another (Figure 4). Our first objective in this section is to define the limit of $f(x,y)$ as (x,y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

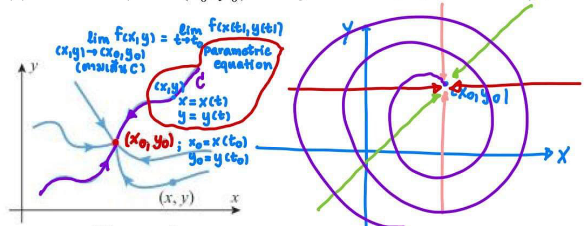


Figure 4

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 918)

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0), y_0 = y(t_0)$ and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (\text{along } C)}} f(x,y) \qquad \text{and} \qquad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0) \\ (\text{along } C)}} f(x,y,z)$$

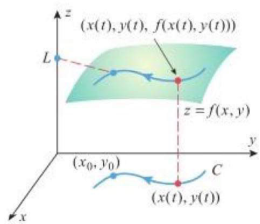
are defined by

$$\boxed{\lim_{\substack{(x,y)\to(x_0,y_0) \\ (\text{along } C)}} f(x,y) = \lim_{t \to t_0} f(x(t), y(t))}$$

$$\boxed{\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0) \\ (\text{along } C)}} f(x,y,z) = \lim_{t \to t_0} f(x(t), y(t), z(t))}$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 5: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x,y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero z-coordinate for points in the xy-plane.

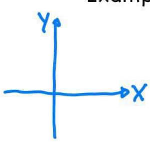


$$\boxed{\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{(along } C)}} f(x,y) = L}$$

Figure 5

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 918)

Example 16 Let $f(x,y) = -\dfrac{xy}{x^2+y^2}$. Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along



(a) the x-axis (b) the y-axis (c) the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$

(a) ตามแกน x:

สมการพาราเมตริก: $x = t$, $y = 0$

$f(x,y) = f(t,0) = 0$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตามแกน x)}}} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0} 0 = 0$

(b) ตามแกน Y:

สมการพาราเมตริก: $x = 0$, $y = t$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตามแกน Y)}}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0} 0 = 0$

(c) ตามเส้น $y = x$:

สมการพาราเมตริก: $x = t$, $y = t$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตาม } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \dfrac{-(t)(t)}{t^2+t^2} = \lim_{t\to 0} \dfrac{-t^2}{2t^2} = -\dfrac{1}{2}$

(d) ตามเส้น $y = -x$:

สมการพาราเมตริก: $x = t$, $y = -t$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตาม } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0} \dfrac{-(t)(-t)}{t^2+(-t)^2} = \lim_{t\to 0} \dfrac{t^2}{2t^2} = \dfrac{1}{2}$

(e) ตามเส้น $y = x^2$

สมการพาราเมตริก: $x = t$, $y = t^2$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตาม } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \dfrac{-(t)(t^2)}{t^2+(t^2)^2} = \lim_{t\to 0} \dfrac{-t^3}{t^2+t^4} = \lim_{t\to 0} \dfrac{-t}{1+t^2} = 0$

6.2 General limits of functions of two variables

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Is intended to convey the idea that the value of $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0).

6.3 Relationships between general limits and limits along smooth curve

ราบเรียบ: ต่อเนื่อง, อนุพันธ์หาค่าได้, อนุพันธ์ต่อเนื่อง

Theorem 1 ★

(a) If $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$, then $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$ along any smooth curve.

(b) If the limit of $f(x,y)$ fails to exist as $(x,y)\to(x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y)\to(x_0,y_0)$.

(a) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \Leftrightarrow \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{(along } C)}} f(x,y) = L$ สำหรับทุก smooth curve C

$P \Rightarrow Q$

$\sim Q \Rightarrow \sim P$

Example 17 The limit $\lim_{(x,y)\to(0,0)} -\dfrac{xy}{x^2+y^2}$ does **not exist** because

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(ตามแกน } x)}} \frac{-xy}{x^2+y^2} = 0 \neq -\frac{1}{2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{(ตาม } y=x)}} \frac{-xy}{x^2+y^2}$$

\#

1 ตัวแปร $f(x)$

ถ้า f ต่อเนื่องที่ $x=x_0$ แล้ว $\lim_{x\to x_0} f(x) = f(x_0)$

2 ตัวแปร

ถ้า f ต่อเนื่องที่ (x_0,y_0) แล้ว $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

6.4 Continuity

Definition 3 A function $f(x,y)$ is said to be **continuous at** (x_0,y_0) if $f(x_0,y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

In addition, if f is continuous at every point in an open set D, then we say that f is **continuous on** D, and if f is continuous at every point in the xy–plane, then we say that f is **continuous everywhere.**

ฟังก์ชันอะไรบ้างที่ต่อเนื่อง ??

Theorem 2

Ex $f(x,y) = x^2 e^y$ ต่อเนื่องใน $\mathbb{R}^2$

$f(x,y) = x\sin y$

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y)=g(x)h(y)$ is continuous at (x_0,y_0).

(b) If $h(x,y)$ is continuous at (x_0,y_0) and $g(u)$ is continuous at $u=h(x_0,y_0)$, then the composition $f(x,y)=g(h(x,y))$ is continuous at (x_0,y_0). $f(x,y) = \sin(x^2y^5)$ ต่อเนื่องใน $\mathbb{R}^2$

(c) If $f(x,y)$ is continuous at (x_0,y_0), and $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0)=x_0$ and $y(t_0)=y_0$, then the composition $f(x(t),y(t))$ is continuous at t_0.

Example 18 Use the Theorem 2 to show that the functions $f(x,y)=3x^2y^5$ and $f(x,y)=\sin(3x^2y^5)$ are continuous everywhere.

$g(x) = 3x^2$ ต่อเนื่องใน $\mathbb{R}$

$h(y) = y^5$ ต่อเนื่องใน $\mathbb{R}$

โดย ⓐ; $f(x,y) = 3x^2y^5 = g(x)h(y)$

ต่อเนื่องทุกจุดใน $\mathbb{R}^2$

$h(x,y) = 3x^2y^5$ ต่อเนื่องใน $\mathbb{R}^2$

$g(u) = \sin u$ ต่อเนื่องใน $\mathbb{R}$

โดย ⓑ; $f(x,y) = \sin(3x^2y^5)$

$= g(h(x,y))$

ต่อเนื่องใน $\mathbb{R}^2$

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example 19 Evaluate $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$

$f(x,y) = \dfrac{xy}{x^2+y^2}$ ต่อเนื่องทุกจุดใน $\mathbb{R}^2$ ยกเว้น $(0,0)$

$\therefore$ $f(x,y)$ ต่อเนื่องที่ $(-1,2)$

$$\lim_{(x,y)\to(-1,2)} f(x,y) = f(-1,2) = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5}$$

Example 20 Since the function $f(x,y)=\dfrac{x^3y^2}{1-xy}$ is a quotient of continuous functions, it is continuous except

$f(x,y)$ ต่อเนื่องทุกจุดยกเว้น $1-xy=0$