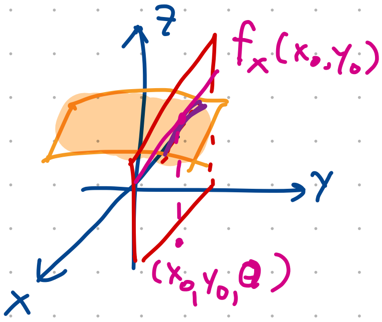
$z = f(x,y)$

หาอนุพันธ์ย่อย $\dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$



ฟังก์ชันตัวแปรเดียว

Thm ถ้า f หาอนุพันธ์ได้ แล้ว f ต่อเนื่อง

ฟังก์ชันหลายตัวแปร

ถ้า f มีอนุพันธ์ย่อย แล้ว f ต่อเนื่อง

จริงไหม?

7.6 Partial derivatives and continuity

Example 28 Let

f มีอนุพันธ์ย่อย $\not\Rightarrow$ f cont

$f(\Delta x, 0) = 0$

$f(0, \Delta y) = 0$

$$f(x,y)=\begin{cases} -\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

(a) Show that $f_x(x,y)$ and $f_y(x,y)$ exist at all points (x,y).

(b) Explain why f is not continuous at (0,0).

(a) สำหรับทุกจุดที่ $(x,y)\neq(0,0)$

$$f_x(x,y) = -\frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2}$$

$$f_y(x,y) = -\frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2}$$

(b) สำหรับจุดที่ $(x,y)=(0,0)$

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{0-0}{\Delta y} = 0$$

$$\therefore f_x(x,y) = \begin{cases} -\dfrac{(x^2+y^2)y - 2x^2y}{(x^2+y^2)^2}; & (x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0) \end{cases}$$

$$f_y(x,y) = \begin{cases} -\dfrac{(x^2+y^2)x - 2xy^2}{(x^2+y^2)^2}; & (x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0) \end{cases}$$

(b) $\boxed{1}$ $f(0,0)=0$

$\boxed{2}$ $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$\therefore$ f ไม่ต่อเนื่องที่ $(0,0)$

7.7 Partial derivatives of functions with more than two variables

For a function $f(x,y,z)$ of three variables, there are three **partial derivatives**:

$$f_x(x,y,z),\quad f_y(x,y,z),\quad f_z(x,y,z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x,y,z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x},\quad \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial w}{\partial z}$$

Example 29 Let $f(x,y,z) = x\sin(y+3z)$. Find f_x, f_y and f_z.

$$f_x = \sin(y+3z)$$

$$f_y = x\cos(y+3z)\frac{\partial}{\partial y}(y+3z) = x\cos(y+3z)$$

$$f_z = x\cos(y+3z)\frac{\partial}{\partial z}(y+3z) = 3x\cos(y+3z)$$

Example 30 If $f(x,y,z)=x^3y^2z^4+2xy+z$, then

$f_x(x,y,z)=$ $3x^2y^2z^4+2y$

$f_y(x,y,z)=$ $2x^3yz^4+2x$

$f_z(x,y,z)=$ $4x^3y^2z^3+1$

$f_z(-1,1,2)=$ $4(-1)^3(1)^2(2)^3+1 = -31$

Example 31 If $f(\rho,\theta,\phi)=\rho^2\cos\phi\sin\theta$, then

$f_\rho(\rho,\theta,\phi)=$ $2\rho\cos\phi\sin\theta$

$f_\theta(\rho,\theta,\phi)=$ $\rho^2\cos\phi\cos\theta$

$f_\phi(\rho,\theta,\phi)=$ $\rho^2(-\sin\phi)\sin\theta$

In general, if $f(v_1,v_2,\ldots,v_n)$ is a function of n variables, there are n partial derivatives of f, each of which is obtained by holding $n-1$ of the variables fixed and differentiating the function f with respect to the remaining variable. If $w=f(v_1,v_2,\ldots,v_n)$, then these partial derivatives are denoted by

$$\frac{\partial w}{\partial v_1},\quad \frac{\partial w}{\partial v_2},\quad \ldots,\quad \frac{\partial w}{\partial v_n}$$

where $\dfrac{\partial w}{\partial v_i}$ is obtained by holding all variables except v_i fixed and differentiating with respect to v_i.

7.8 Higher – order partial derivatives

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible **second–order partial derivatives** of f, which are defined by

$$f(x,y)$$
$$f_x \quad f_y$$
$$f_{xx} \quad f_{xy} \quad f_{yx} \quad f_{yy}$$
$$f_{xxx} \quad f_{xxy} \quad f_{xyx} \quad f_{xyy}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

mixed second-order partial derivative "อนุพันธ์ย่อยผสม"

The last two cases are called the **mixed second-order partial derivatives** or the **mixed second partials**. Also, the derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are often called the **first-order partial derivatives** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

Example 32 Find the second-order partial derivatives of $f(x,y) = x^2y^3 + x^4y$.

$f_x = 2xy^3 + 4x^3y$

$f_y = 3x^2y^2 + x^4$

$f_{xx} = 2y^3 + 12x^2y$

$f_{yy} = 6x^2y$

$f_{xy} = 6xy^2 + 4x^3$

$f_{yx} = 6xy^2 + 4x^3$

โดยทั่วไป f_{xy} , f_{yx} อาจจะเท่าหรือไม่เท่าก็ได้

Thm ถ้า $f(x,y)$ และอนุพันธ์ย่อย f_x, f_y, f_{yx} นิยามบนบริเวณเปิดที่มีจุด (a,b) และทั้งหมดมีความต่อเนื่องที่จุด (a,b) แล้ว $f_{xy}(a,b) = f_{yx}(a,b)$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example 33 Let $f(x,y) = y^2e^x + y$. Find f_{xyy}.

$f_x = y^2e^x$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = 2ye^x$

$\frac{\partial^3 f}{\partial y^2 \partial x}$ $f_{xyy} = 2e^x$

8. The Chain rule

8.1 Chain rules for derivatives

Theorem 3 (Chain rules for derivatives)

If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

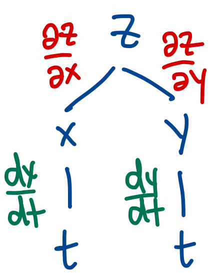
where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t), y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Example 34 Suppose that $z = x^2 y, \quad , x = t^2, \quad y = t^3$. Use the chain rule to find $\frac{dz}{dt}$, and check the result by expressing z as a function of t and differentiating directly.



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= (2xy)\cdot(2t) + (x^2)(3t^2)$$
$$= 2\cdot t^2 \cdot t^3 \cdot 2t + t^4 \cdot 3t^2$$
$$= 7t^6$$

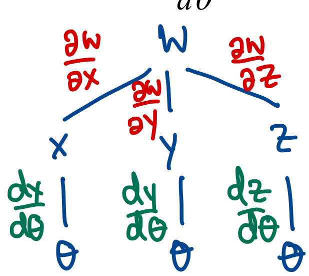
$z = x^2 y \; ; x = t^2, \; y = t^3$

$\downarrow$

$z = (t^2)^2 \cdot t^3 = t^7$

$\frac{dz}{dt} = 7t^6$

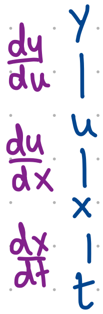
Example 35 Suppose that $w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta.$ Use the chain rule to find $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{4}$.



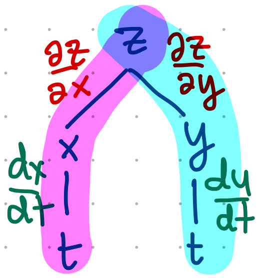
$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\cdot\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\cdot\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\cdot\frac{dz}{d\theta}$$
$$= \cancel{\frac{2x}{2\sqrt{x^2+y^2+z^2}}(-\sin\theta)} + \cancel{\frac{2y}{2\sqrt{x^2+y^2+z^2}}(\cos\theta)} + \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}}\sec^2\theta$$
$$= \frac{\tan\theta \cdot \sec^2\theta}{\sqrt{\sin^2\theta + \cos^2\theta + \tan^2\theta}} = \frac{\tan\theta \cdot \sec^2\theta}{\sqrt{1 + \tan^2\theta}}$$
$$= \frac{\tan\theta \cdot \sec^2\theta}{\sqrt{\sec^2\theta}} = \tan\theta \sec\theta$$

[illegible] $\left.\frac{dw}{d\theta}\right|_{\theta = \frac{\pi}{4}} = \sqrt{2}$ #

$y=f(u)\ ,\ u=u(x),\ x=x(t)$ หา $\dfrac{dy}{dt}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}\cdot\dfrac{dx}{dt}$

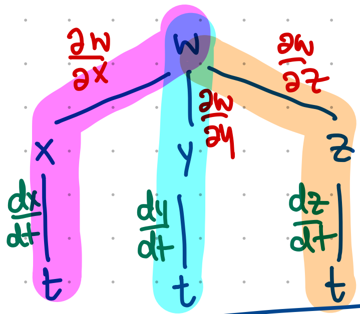


$z=f(x,y)\ ,\ x=x(t)\ ,\ y=y(t)$ หา $\dfrac{dz}{dt}$

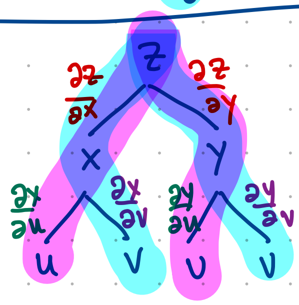


$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

$w=f(x,y,z)\ ,\ x=x(t)\ ,\ y=y(t)\ ,\ z=z(t)$ หา $\dfrac{dw}{dt}$



$$\boxed{\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt}}$$

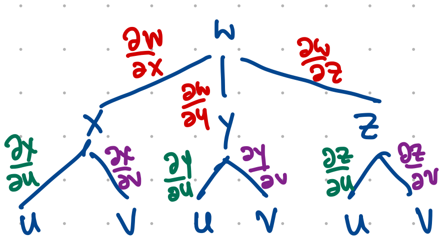


$z=f(x,y),\ x=x(u,v),\ y=y(u,v)$

$\dfrac{\partial z}{\partial u}\ ,\ \dfrac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



$w=f(x,y,z),\ x=x(u,v),\ y=y(u,v), z=z(u,v)$

หา $\dfrac{\partial w}{\partial u}\ ,\ \dfrac{\partial w}{\partial v}$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

8.2 Chain rules for partial derivatives

Theorem 4 (Chain rules for partial derivatives)

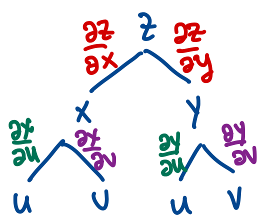
If $x = x(u,v)$ and $y = y(u,v)$ have first-order partial derivatives at the point (u,v), and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(u,v), y(u,v))$, then $z = f(x(u,v), y(u,v))$ has first-order partial derivatives at the point (u,v) given by

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

If $x = x(u,v), y = y(u,v)$ and $z = z(u,v)$ have first-order partial derivatives at the point (u,v), and if the function $w = f(x,y,z)$ is differentiable at the point $(x,y,z) = (x(u,v), y(u,v), z(u,v))$, then $z = f(x(u,v), y(u,v), z(u,v))$ has first-order partial derivatives at the point (u,v) given by

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}}$$

Example 36 Given that $z = e^{xy}, \quad ,x = 2u + v, \quad y = \frac{u}{v}$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the chain rule.



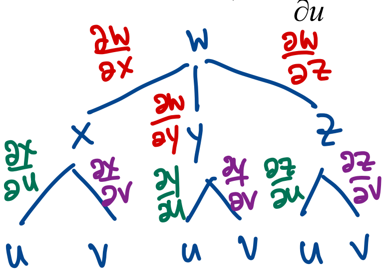
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right) \qquad \#$$

Example 36 Suppose that $w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$. Use appropriate forms of the chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= (yze^{xyz})(3) + (xze^{xyz})(3) + (xye^{xyz})(2uv)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= (yze^{xyz})(1) + (xze^{xyz})(-1) + (xye^{xyz})(u^2) \qquad \#$$