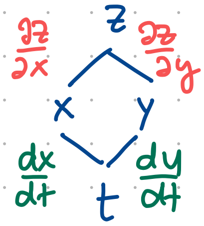
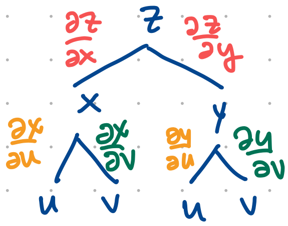
$z = f(x,y)$, $x = x(t)$, $y = y(t)$ หา $\frac{dz}{dt}$



$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$ หา $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

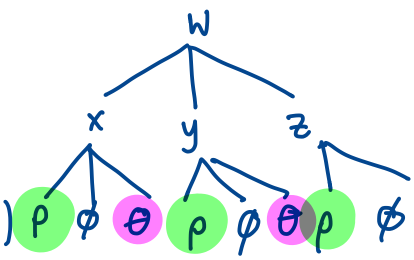
8.3 Other versions of the chain rule

Although we will not prove it, the chain rule extends to functions $w = f(v_1, v_2, \ldots, v_n)$ of n variables. For example, if each v_i is a function of t, $i = 1, 2, \ldots, n$, the relevant formula is

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt}}$$

Example 37 Suppose that $w = x^2 + y^2 - z^2$, $\quad x = \rho \sin\phi \cos\theta$, $\quad y = \rho \sin\phi \sin\theta$, $\quad z = \rho \cos\phi$.

Use appropriate forms of the chain rule to find $\dfrac{\partial w}{\partial \rho}$ and $\dfrac{\partial w}{\partial \theta}$.



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$

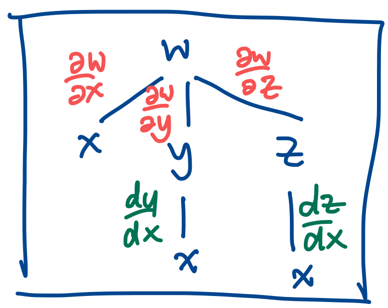
$$= (2x)\cdot(\sin\phi\cos\theta) + (2y)\cdot(\sin\phi\sin\theta) + (-2z)(\cos\phi)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= (2x)\cdot(-\rho\sin\phi\sin\theta) + (2y)(\rho\sin\phi\cos\theta) \quad \#$$

Example 38 Suppose that $w = xy + yz$, $\quad y = \sin x$, $\quad z = e^x$.

Use appropriate forms of the chain rule to find $\dfrac{dw}{dx}$.



$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dx}$$

$$= y + (x+z)\cos x + y e^x \quad \#$$

8.4 Implicit differentiation

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Then

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad (*)$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \qquad (**)$$

defines y implicitly as a differentiable function of x and we are interested in finding $\frac{dy}{dx}$. Differentiating both sides of (**) with respect to x and applying (*) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

Thus, if $\frac{\partial f}{\partial y} \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}.$$

In summary, we have the following result.

Theorem 5 If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\boxed{\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}}$$

Example 39 Given that $x^3 + y^2 x - 3 = 0.$

find $\frac{dy}{dx}$ using Theorem 5 and check the result using implicit differentiation.

Implicit function

$y = x^2$ $\boxed{y=f(x)}$ explicit function

$xy - \sin(xy) = x^2y^2$ ← implicit function

Ex 39 $\quad \underbrace{x^3 + y^2x - 3}_{f(x,y)} = 0$

$$\frac{d}{dx}[x^3+y^2x-3] = \frac{d}{dx}[0]$$

$$3x^2 + y^2 + x(2y)\frac{dy}{dx} = 0$$

วิธีใน Cal 1 $\quad \therefore \frac{dy}{dx} = -\frac{3x^2+y^2}{2xy}$

อีกวิธี ให้ $f(x,y) = x^3+y^2x-3 = 0$

$$\therefore f_x = 3x^2+y^2$$

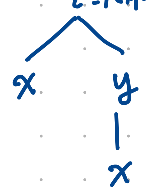
$$f_y = 2xy$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2+y^2}{2xy}$$ วิธีใน Cal 2

→ $z = f(x,y) = x^3+y^2x-3 = 0$ หา $\frac{dy}{dx}$

โดยที่ $y = g(x)$

$z = f(x,y)$ หา $\frac{dz}{dx}$

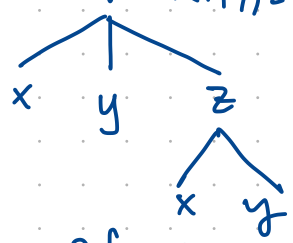


$$\boxed{0 = \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}}$$

$$\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = -\frac{\partial f}{\partial x}$$

$$\boxed{\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}}$$

CASE 2 $\quad w = f(x,y,z)$ โดยที่ $z = z(x,y)$ → $w = f(x,y,z)$



Ex 40 $\quad x^2+y^2+z^2 = 1$, หา $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ ที่ $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

(Ex 27 หา $\frac{\partial z}{\partial y}$)

$$\frac{\partial}{\partial y}(x^2+y^2+z^2) = \frac{\partial}{\partial y}(1)$$

$$2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{2y}{2z}$$

โดยใช้กฎลูกโซ่ สร้าง $f(x,y,z) = x^2+y^2+z^2-1$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)} = -\frac{\frac{2}{3}}{\frac{2}{3}} = -1$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{2y}{2z} = -\frac{y}{z}$$

$$\therefore \left.\frac{\partial z}{\partial y}\right|_{\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)} = -\frac{\frac{1}{3}}{\frac{2}{3}} = -\frac{1}{2}$$

- $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$ หา $\frac{\partial z}{\partial x}$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{f_x}{f_z}}$$

- $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$

$$0 = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$\boxed{\therefore \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{f_y}{f_z}}$$

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of x, y, and z and z is a differentiable function of x and y. It follows from Theorem 5 that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}. \qquad (***)$$

If the equation

$$f(x, y, z) = c \qquad (****)$$

defines z implicitly as a differentiable function of x and y, then taking the partial derivative of each side of (****) with respect to x and applying (***) gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0.$$

If $\frac{\partial f}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} = -\frac{f_x}{f_z}.$$

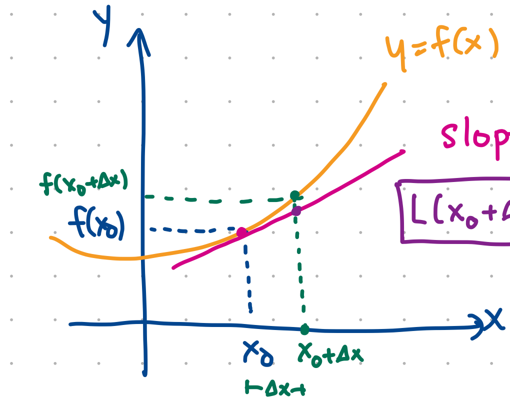
A similar result holds for $\frac{\partial z}{\partial y}$.

Theorem 6 If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\frac{\partial f}{\partial z} \neq 0$, then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} = -\frac{f_x}{f_z}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z} = -\frac{f_y}{f_z}}.$$

Example 40 Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Differentials and Local Linearity



slope $= f'(x_0)$

$$\boxed{L(x_0+\Delta x) = f(x_0) + f'(x_0)\,\Delta x}$$

ถ้า Δx น้อยๆ เราประมาณ

$$\underline{f(x_0+\Delta x)} \approx \underline{L(x_0+\Delta x)}$$

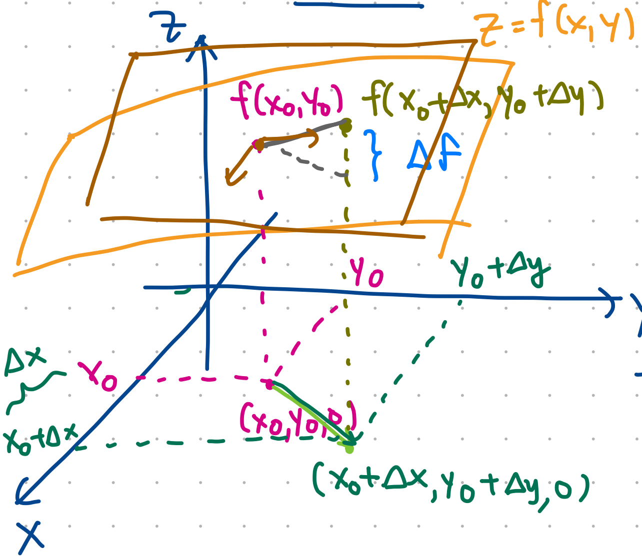
$$\Delta f = f(x_0+\Delta x) - L(x_0+\Delta x)$$

<u>differential</u>: $\boxed{dy = f'(x)dx}$

ถ้าให้ $dx = \Delta x$ จะได้ว่า

$$\underset{(\Delta f)}{\Delta y} \approx dy = f'(x)dx$$

กรณีฟังก์ชันสองตัวแปร



ส่วนเปลี่ยนแปลง : Incremental

$$\Delta f = \underline{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)} \quad — (*)$$

ระนาบสัมผัส ที่จุด (x_0, y_0) ตามแนวความชัน f_x, f_y.

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$\boxed{\Delta f \approx f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y} \quad — ♡$$

$(\Delta f = L(x,y) - f(x_0,y_0).)$

ดิฟเฟอเรนเชียลรวม
(total differential at (x_0,y_0))

$$\boxed{dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy} \quad — ♡♡$$

ถ้าเราประมาณ $\Delta f \approx df$

∴ เราประมาณ $\boxed{\Delta z \approx f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.}$

9. Differentials and local linearity

9.1 Differentials

As with the one-variable case, the approximations

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$\boxed{dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the **total differential of z at (x_0, y_0)** or as the **total differential of f at (x_0, y_0)**. Similarly, for a function $w = f(x, y, z)$ of three variables we have the **total differential of w at (x_0, y_0, z_0)**,

$$\boxed{dw = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz}$$

which is also referred to as the **total differential of f at (x_0, y_0, z_0)**. It is common practice to omit the subscripts and write as

$$\boxed{dz = f_x(x, y)dx + f_y(x, y)dy}$$

and

$$\boxed{dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz}$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form

$$\boxed{\Delta f \approx df}$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation $\Delta f \approx df$ as

$$\boxed{\Delta z \approx dz} \qquad (*)$$

In other words, we can estimate the change Δz in z by the value of the differential dz where dx is the change in x and dy is the change in y. Furthermore, if Δx and Δy are close to 0, then the magnitude of the error in approximation (*) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$.

Example 41 Use (*) to approximate the change in $z = xy^2$ from its value at (0.5, 1.0) to its value at (0.503, 1.004). Compare the magnitude of the error in this approximation with the distance between the points (0.5, 1.0) and (0.503, 1.004).

$z = f(x,y) \to f(0.5, 1.0)$

$f(0.503, 1.004)$

$dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$

$f_x(x,y) = y^2$

$f_y(x,y) = 2xy$

$\Big\}\ dz = (y^2)dx + (2xy)dy$

ที่ $(0.5, 1)$; $dz = (1)^2 dx + 2(0.5)(1)dy = dx + dy$

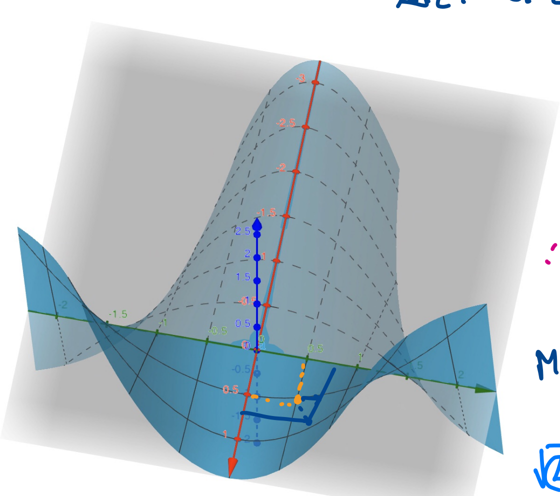
ถ้าขยับไปที่จุด$(0.503, 1.004)$; $dx = \Delta x = 0.003$, $dy = \Delta y = 0.004$

$\Delta z \approx dz = 0.003 + 0.004 = 0.007$

จาก

$\dfrac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \dfrac{0.00032}{0.005} = 0.0064$

$< \dfrac{1}{150}$

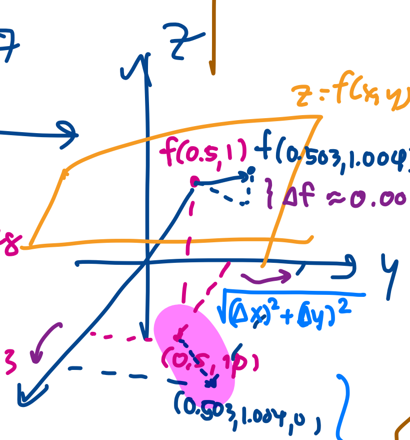


กดเครื่องคิดเลข

$f(0.503, 1.004) = 0.507032048$

$\therefore \Delta f = f(0.503, 1.004) - f(0.5, 1)$

$= 0.50703\ldots - 0.5 = 0.00703$



Magnitude of error:

$|dz - \Delta z| = |0.007 - 0.00703| \approx 0.00032$

$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = 0.005$

9.2 Local linear approximations

We now show that if a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x,y)$ is differentiable at the point (x_0, y_0). Then approximation can be written in the form

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

If we let $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, this approximation becomes

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \qquad (**)$$

which yields a linear approximation of $f(x,y)$. Since the error in this approximation is equal to the error in approximation, we conclude that for (x,y) close to (x_0,y_0),the error in (**) will be much smaller than the distance between these two points. When $f(x,y)$ is differentiable at (x_0,y_0) we get

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

and refer to $L(x,y)$ as **the local linear approximation to f at (x_0,y_0).**

Example 42 Let $L(x,y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2+y^2}$ at the point (3, 4). Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points (3, 4) and (3.04, 3.98).