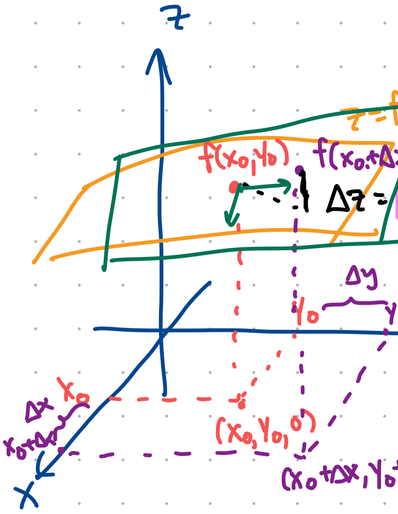


$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

Local linear approximation

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

ถ้า Δx, Δy เล็กๆ สามารถประมาณ

$$\Delta f \approx df$$

(ทราบว่า $\boxed{df = f_x(x,y)\,dx + f_y(x,y)\,dy}$)

Total differential.

9.2 Local linear approximations

We now show that if a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x,y)$ is differentiable at the point (x_0, y_0). Then approximation can be written in the form

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

If we let $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, this approximation becomes

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \qquad (**)$$

which yields a linear approximation of $f(x,y)$. Since the error in this approximation is equal to the error in approximation, we conclude that for (x,y) close to (x_0,y_0),the error in (**) will be much smaller than the distance between these two points. When $f(x,y)$ is differentiable at (x_0,y_0) we get

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

and refer to $L(x,y)$ as **the local linear approximation to f at (x_0,y_0).**

Example 42 Let $L(x,y)$ denote the local linear approximation to $f(x,y)=\sqrt{x^2+y^2}$ at the point (3, 4). Compare the error in approximating

(กดเครื่องคิดเลข)

$$f(3.04, 3.98) = \sqrt{(3.04)^2+(3.98)^2} \approx 5.00819$$

by $L(3.04, 3.98)$ with the distance between the points (3, 4) and (3.04, 3.98).

หาค่าของ $L(3.04, 3.98)$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f_x(x,y) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

เลือกจุดใกล้ๆ คือ (3,4)

$f_x(3,4) = \frac{3}{5}$

$f_y(3,4) = \frac{4}{5}$

$$\therefore L(3.04, 3.98) = f(3,4) + f_x(3,4)(3.04-3) + f_y(3,4)(3.98-4)$$

$$= 5 + \left(\tfrac{3}{5}\right)(0.04) + \left(\tfrac{4}{5}\right)(-0.02)$$

$$= 5.008$$

ค่าจริง — ค่าประมาณ

error: $|f(3.04, 3.98) - L(3.04, 3.98)| = |5.00819 - 5.008| = 0.00019$

distance $(3,4)$ and $(3.04, 0.98) = \sqrt{(0.04)^2 + (-0.02)^2} \approx 0.045$

$$\therefore \frac{|5.00819 - 5.008|}{\sqrt{(0.04)^2+(-0.02)^2}} \approx 0.042 < \frac{1}{200}$$

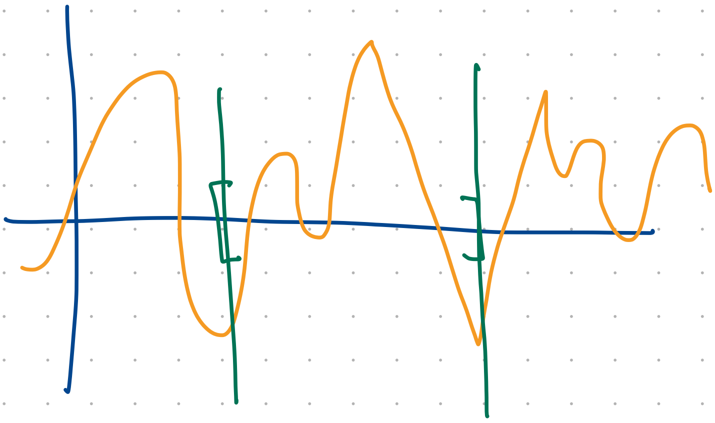
$\boxed{206111}$

extremum
- Maximum
- Minimum

Relative extremum ค่าสุดขีดสัมพัทธ์ **VS** **Absolute extremum** ค่าสุดขีดสัมบูรณ์

สูงสุด/ต่ำสุดเมื่อเทียบกับบริเวณรอบๆ

สูงสุด/ต่ำสุด ในบริเวณที่กำหนด



การหาค่าสุดขีดสัมพัทธ์ของ $y=f(x)$

① หาจุดวิกฤต (critical point) $x=c$ เป็นจุดวิกฤต ถ้า $f'(c)=0$ หรือ $f'(c)$ หาค่าไม่ได้
โดย c ต้องอยู่ในโดเมน

② เอาจุดวิกฤตไปทดสอบ

$\boxed{1}$ ทดสอบโดยอนุพันธ์อันดับหนึ่ง

+ + + | − − − Rel. Max
$f(x)>0$ c $f'(x)<0$

− − − | + + + Rel. Min
$f'(x)<0$ c $f'(x)>0$

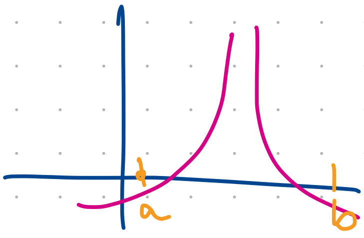
$\boxed{2}$ ทดสอบโดยอนุพันธ์อันดับสอง

$f''(c)>0 \Rightarrow x=c$ ให้ค่าต่ำสุดสัมพัทธ์
$f''(c)<0 \Rightarrow x=c$ ให้ค่าสูงสุดสัมพัทธ์
$f''(c)=0 \Rightarrow$ สรุปไม่ได้

การหาค่าสุดขีดสัมบูรณ์ของ $y=f(x)$ บน $[a,b]$

① หาจุดวิกฤต $x=c$

② เทียบ $f(a)$, $f(b)$, $f(c)$



“optimization” • Hessian matrix

10. Maxima and minima of functions of two variables

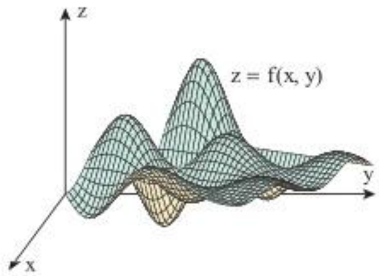
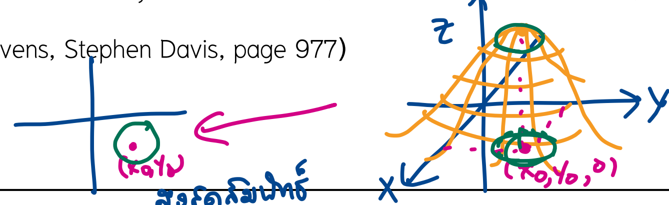


Figure 9

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 977)



10.1 Extrema ค่าสุดขีด

Definition 5 A function f of two variables is said to have a **relative maximum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

Definition 6 A function f of two variables is said to have a **relative minimum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a **relative extremum** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an **absolute extremum** at (x_0, y_0).

Figure 10 shows the graph of a function f whose domain is the square region in the xy-plane whose points satisfy the inequalities $0 \leq x \leq 1, 0 \leq y \leq 1$. The function f has relative minima at the points A and C and a relative maximum at B. There is an absolute minimum at A and an absolute maximum at D.

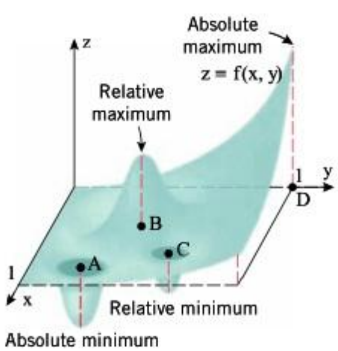
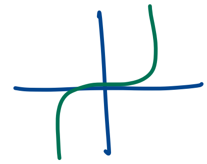


Figure 10

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 977)

Thm : x_0 ให้ค่าสุดขีดสัมพัทธ์ $f'(x_0)$ หาค่าได้ $\Rightarrow f'(x_0) = 0$



10.2 Finding relative extrema

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0)=0$. To obtain the analog of this result for functions of two variables, suppose that $f(x,y)$ has a relative maximum at a point (x_0,y_0) and that the partial derivatives of f exist at (x_0,y_0). It seems plausible geometrically that the traces of the surface $z=f(x,y)$ on the planes $x=x_0$ and $y=y_0$ have horizontal tangent lines at (x_0,y_0) (Figure 11), so

$$f_x(x_0,y_0)=0 \qquad \text{and} \qquad f_y(x_0,y_0)=0.$$

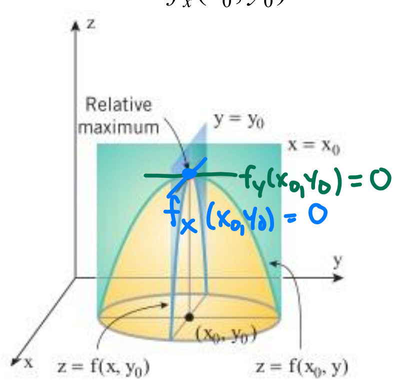


Figure 11

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 979)

The same conclusion holds if f has a relative minimum at (x_0,y_0), all of which suggests the following result, which we state without formal proof.

Theorem 7 If f has a relative extremum at a point (x_0,y_0), and if the first-order partial derivatives of f exist at this point, then

$$f_x(x_0,y_0)=0 \qquad \text{and} \qquad f_y(x_0,y_0)=0.$$

บทกลับจริงไหม? $f_x(x_0,y_0)=0=f_y(x_0,y_0) \overset{?}{\Longrightarrow} f$ ให้ค่า extremum ที่ (x_0,y_0)

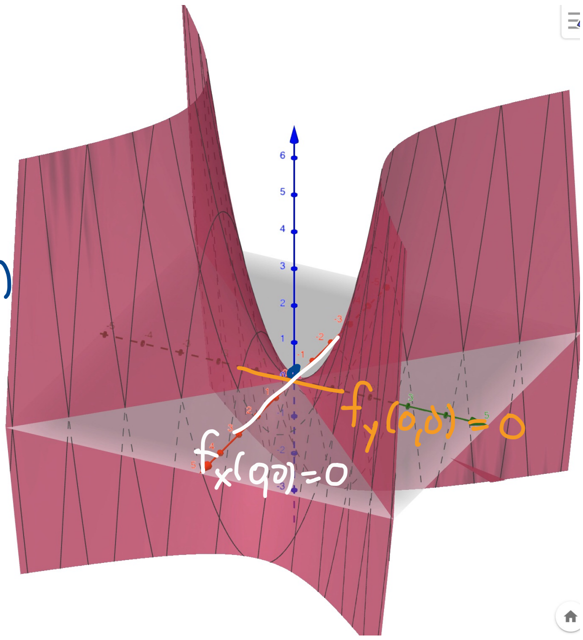
Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x)=0$ or f is not differentiable. The following definition is the analog for functions of two variables.

Definition 7 A point (x_0,y_0) in the domain of a function $f(x,y)$ is called a **critical point** of the function if $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$ or if one or both partial derivatives do not exist at (x_0,y_0).

บทกลับจริงไหม? $f_x(x_0,y_0)=0=f_y(x_0,y_0) \overset{?}{\Longrightarrow}$ f ให้ค่า extremum ที่ (x_0,y_0)

ต.ย.ค้าน hyperbolic paraboloid

$$f(x,y) = y^2 - x^2$$



หาจุดวิกฤต $\underbrace{\begin{aligned} f_x(x,y) &= -2x \\ f_y(x,y) &= 2y \end{aligned}}$ } หา (x_0,y_0) ที่ทำให้ $f_x(x_0,y_0)=0=f_y(x_0,y_0)$

หรือ

$f_y(x_0,y_0)$, $f_x(x_0,y_0)$ หาค่าไม่ได้

$\downarrow$

Set $\underbrace{\begin{aligned} f_x(x,y) &= -2x = 0 \Rightarrow x=0 \\ f_y(x,y) &= 2y = 0 \Rightarrow y=0 \end{aligned}}$

$(x,y) = (0,0)$ ไม่ให้ค่าสูงสุด / ต่ำสุดสัมพัทธ์

“จุดอานม้า (saddle point)

It follows from this definition and Theorem 7 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at every critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 12). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

$$f(x,y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in Figure , has a critical point at $(0,0)$, since

$$f_x(x,y) = -2x \qquad \text{and} \qquad f_y(x,y) = 2y$$

from which it follows that

$$f_x(0,0) = 0 \qquad \text{and} \qquad f_y(0,0) = 0.$$

However, the function f has neither a relative maximum nor a relative minimum at $(0,0)$.
For obvious reasons, the point $(0,0)$ is called a saddle point of f. In general, we will say that a surface $z = f(x,y)$ has a **saddle point** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

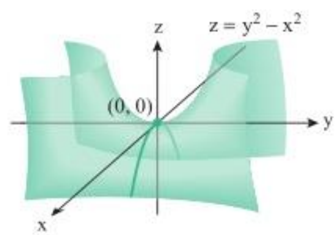
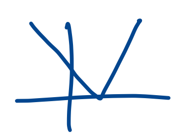


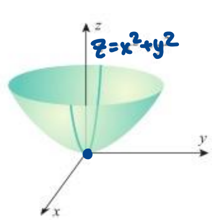
Figure 12

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 979)

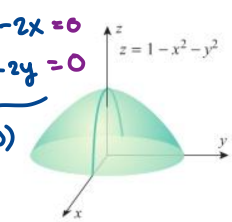
Example 43 The three functions graphed in Figure all have critical points at $(0,0)$.



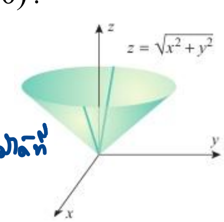
$f_x(x,y) = 2x = 0$
$f_y(x,y) = 2y = 0$
$x=0, y=0$



$f_x(x,y) = -2x = 0$
$\underbrace{f_y(x,y) = -2y = 0}$
$(x,y) = (0,0)$



$f_x = \dfrac{x}{\sqrt{x^2+y^2}}$
$f_y = \dfrac{y}{\sqrt{x^2+y^2}}$
อนุพันธ์ย่อยหาค่าไม่ได้ที่ $(0,0)$



$f_x(0,0) = f_y(0,0) = 0$	$f_x(0,0) = f_y(0,0) = 0$	$f_x(0,0)$ and $f_y(0,0)$ do not exist
relative and absolute min at $(0,0)$	relative and absolute max at $(0,0)$	relative and absolute min at $(0,0)$

Figure 13

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 980)

Discriminant $\rightarrow$ $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$

STEP
① หาจุดวิกฤต
② เอาจุดวิกฤตมาทดสอบ

10.3 The second partial test

Theorem 8 (The Second Partials Test) Let f be a function of two variables with continuous second–order partial derivatives in some disk centered at a critical point (x_0, y_0), and let

$$D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}^{\ 2}(x_0,y_0)$$

(a) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a relative minimum at (x_0,y_0).

(b) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a relative maximum at (x_0,y_0).

(c) If $D < 0$, then f has a saddle point at (x_0,y_0).

(d) If $D = 0$, then no conclusion can be drawn.

$\boxed{D}$ @(x_0,y_0)

- $\boxed{D>0}$
  - $f_{xx}(x_0,y_0)>0$: ที่ (x_0,y_0) จะให้ค่าต่ำสุดสัมพัทธ์
  - $f_{xx}(x_0,y_0)<0$: ที่ (x_0,y_0) จะให้ค่าสูงสุดสัมพัทธ์
- $\boxed{D<0}$: ที่ (x_0,y_0) จะเป็นจุดอานม้า
- $\boxed{D=0}$: สรุปไม่ได้

Example 44 Locate all relative extrema and saddle points of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$.

① หาจุดวิกฤต $f_x(x,y) = 6x - 2y = 0$ —①

$f_y(x,y) = -2x + 2y - 8 = 0$ —②

①+②; $4x = 8 \Leftrightarrow x = 2$

$\therefore y = 6$

$\therefore (x,y) = (2,6)$ เป็นจุดวิกฤต

ทด

$f_{xx} = 6$

$f_{yy} = 2$

$f_{xy} = -2$

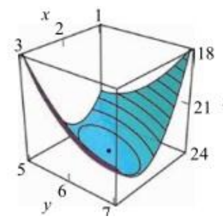


Figure Example 44

② ตรวจสอบโดยใช้ Discriminant ที่จุดวิกฤต

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} = 12 - (+4) = 8 > 0$$

Check $f_{xx} = 6 > 0$ แสดงว่า ที่จุด $(2,6)$ ให้ค่าต่ำสุดสัมพัทธ์ โดยที่ $f(2,6)$ เป็นค่าต่ำสุดสัมพัทธ์ #

Example 45 Locate all relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$.

① หาจุดวิกฤต $f_x(x,y) = 4y - 4x^3 = 0$ —①

$f_y(x,y) = 4x - 4y^3 = 0$ —②

①; $y = x^3$ —③

②; $x = y^3$ —④

แทน ③ ใน ④; $x = (x^3)^3 \Leftrightarrow x = x^9$

$\therefore x^9 - x = 0$

$x(x^8 - 1) = 0$

$x(x^4-1)(x^4+1) = 0$

$x(x^2-1)(x^2+1)(x^4+1) = 0$

$x(x-1)(x+1)(x^2+1)(x^4+1) = 0$

$\therefore x = 0, 1, -1$

$\therefore$ จุดวิกฤต คือ $(0,0), (1,1), (-1,-1)$

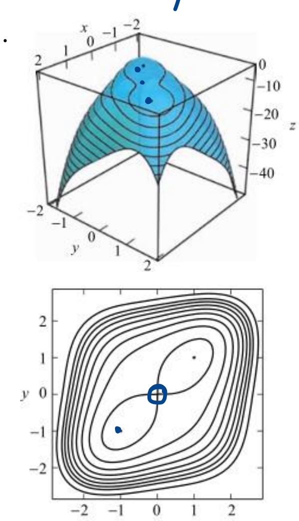


Figure Example 45

② ทดสอบจุดวิกฤต

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$

$f_{xy} = 4$

$\boxed{\text{จุด } (0,0)}$ $D = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0 \Leftrightarrow$ ที่ $(0,0)$ เป็นจุดอานม้า

$\boxed{\text{จุด } (1,1)}$ $D = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$ check $f_{xx} = -12 < 0$, ที่ $(1,1)$ ให้ค่าสูงสุดสัมพัทธ์

$\boxed{\text{จุด } (-1,-1)}$ $D = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$, $f_{xx} = -12 < 0$ ที่ $(-1,-1)$ ให้ค่าสูงสุดสัมพัทธ์ #