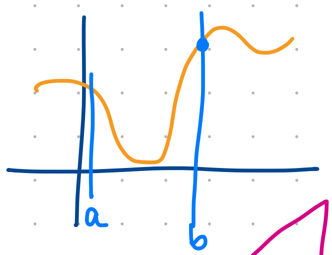
$z = f(x,y)$

1. จุดวิกฤต (a,b) เป็นจุดวิกฤต ถ้า $f_x(a,b) = 0$, $f_y(a,b) = 0$
2. ทดสอบจุดวิกฤต $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

$\boxed{D}$ @ (a,b)

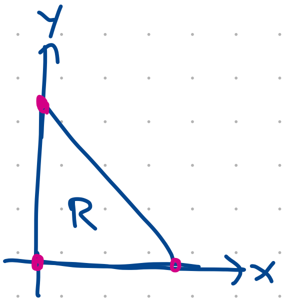
- $\boxed{D>0}$
  - $f_{xx}(a,b) > 0$ ให้ค่า Rel. Min
  - $f_{xx}(a,b) < 0$ ให้ค่า Rel. Max
- $\boxed{D<0}$ จุดอานม้า
- $\boxed{D=0}$ สรุปไม่ได้

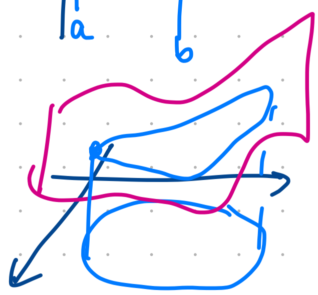
ค่าสุดขีดสัมบูรณ์ : ดูบนบริเวณปิดที่มีขอบเขต closed and bounded region



1. หาจุดที่เป็นจุดวิกฤต
2. จุดที่ขอบ

"เอาทุกคนมาแข่งกัน"





$f(a,b)$ มากสุด → ค่าสูงสุดสัมบูรณ์
น้อยสุด → ค่าต่ำสุดสัมบูรณ์

10.4 Finding absolute extrema on closed and bounded sets

> ***How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R***
>
> **Step 1.** Find the critical points of f that lie in the interior of R.
>
> **Step 2.** Find all boundary points at which the absolute extrema can occur.
>
> **Step 3.** Evaluate $f(x,y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example 46 Find the absolute maximum and minimum values of $f(x,y)=3xy-6x-3y+7$ on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

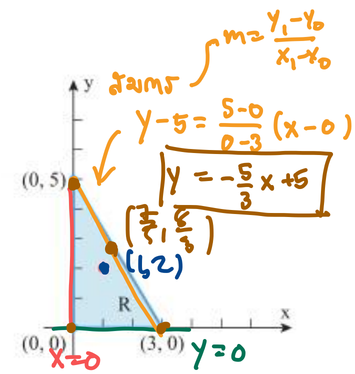


Figure Example 46

① จุดวิกฤตในบริเวณที่พิจารณา

$f_x = 3y-6 = 0$

$f_y = 3x-3 = 0$

$\therefore\ y=2,\ x=1 \rightarrow$ จุดวิกฤต คือ $(1,2)$

② จุดยอดของรูปสามเหลี่ยม $(0,0), (3,0), (0,5)$

③ จุดที่อยู่บนเส้นขอบต่างๆ

เส้นตรง $x=0$ $(0\le y\le 5)$: $u(y)=f(0,y) = -3y+7$

หา $u'(y) = -3 \neq 0 \leftarrow$ บน $x=0$ ไม่เกิดค่าสุดขีดสัมพัทธ์

เส้นตรง $y=0$ $(0\le x\le 3)$: $v(x)=f(x,0) = -6x+7$

หา $v'(x) = -6 \neq 0 \leftarrow$ บน $y=0$ ไม่เกิดค่าสุดขีดสัมพัทธ์

เส้นตรง $y=-\frac{5}{3}x+5$: $w(x)=f(x,-\frac{5}{3}x+5) = 3x(-\frac{5}{3}x+5)-6x-3(-\frac{5}{3}x+5)+7$

$\therefore\ w(x) = -5x^2+15x-6x+5x-15+7$

$w(x) = -5x^2+14x-8$

หา $w'(x) = -10x+14 = 0$

$\therefore\ x=\frac{7}{5} \Longleftrightarrow y = -\frac{\cancel{5}}{3}\cdot\frac{7}{\cancel{5}}+5 = \frac{8}{3}$

จุดบนเส้น $y=-\frac{5}{3}x+5$ คือ $(\frac{7}{5},\frac{8}{3})$

พิจารณาค่า (x,y)	$(1,2)$	$(0,0)$	$(3,0)$	$(0,5)$	$(\frac{7}{5},\frac{8}{3})$
$f(x,y)$	1	7	−11	−8	9/5
		Abs. Max	Abs. Min		

สรุป f บนบริเวณดังกล่าวจะให้
- ค่าสูงสุดสัมบูรณ์ที่ $(0,0)$ มีค่าเป็น 7
- ค่าต่ำสุดสัมบูรณ์ที่ $(3,0)$ มีค่าเป็น −11

$$V = xyz \iff 32 = xyz \iff z = \frac{32}{xy}$$

Example 47 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

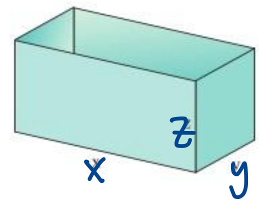


Figure Example 47

Minimize $S = xy + 2yz + 2xz$

$$= xy + 2\cancel{y}\left(\frac{32}{x\cancel{y}}\right) + 2\cancel{x}\left(\frac{32}{\cancel{x}y}\right)$$

$$\therefore \boxed{S = xy + \frac{64}{x} + \frac{64}{y}}$$

หาจุดวิกฤต $S_x = y - \frac{64}{x^2} = 0$ ①

$S_y = x - \frac{64}{y^2} = 0$ ②

แทน ① ใน ②

$$\therefore y = \frac{64}{x^2} \implies x - \frac{64}{\left(\frac{64}{x^2}\right)^2} = 0$$

$$x - \frac{x^4}{64} = 0$$

$$64x - x^4 = 0$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0$$

ถ้า $x = 0$ ไม่เกิดกล่อง

$$x = \cancel{0}, 4$$

$$\therefore y = 4$$

จุดวิกฤตคือ $(4,4)$

ทดสอบจุดวิกฤต $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} \frac{128}{x^3} & 1 \\ 1 & \frac{128}{y^3} \end{vmatrix}$

ที่ $(4,4)$; $D = \begin{vmatrix} \frac{128}{64} & 1 \\ 1 & \frac{128}{64} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 > 0$

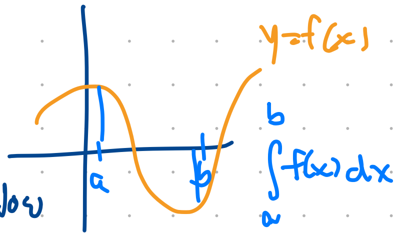
$$f_{xx}(4,4) = 1 > 0$$

$\therefore$ ที่จุด $(4,4)$ ให้ค่าต่ำสุดสัมพัทธ์ $\implies 32 = 4 \cdot 4(z)$

$$\boxed{z = 2}$$

$\therefore$ กล่องที่มีปริมาตร 32 ลบ.ฟุต จะใช้วัสดุทำกล่องน้อยที่สุด ถ้ากล่องมีความกว้าง 4 ฟุต ยาว 4 ฟุต สูง 2 ฟุต #

CHAPTER 3 Multiple Integrals



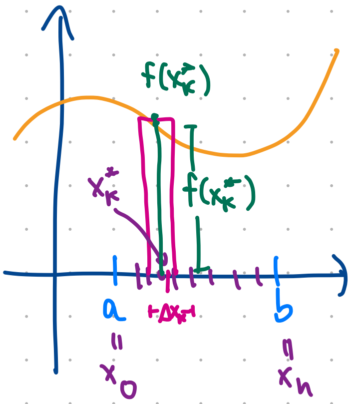
หา $\int_a^b f(x)dx$ $\Rightarrow$ ① แบ่ง $[a,b]$ ออกเป็น n ช่วงย่อย

ความกว้างของช่วงย่อย Δx_k (ช่วงที่ k)

$[x_{k-1}, x_k]$

② ความสูง : เลือกจุด x_k^* ใดๆ ใน $[x_{k-1}, x_k]$

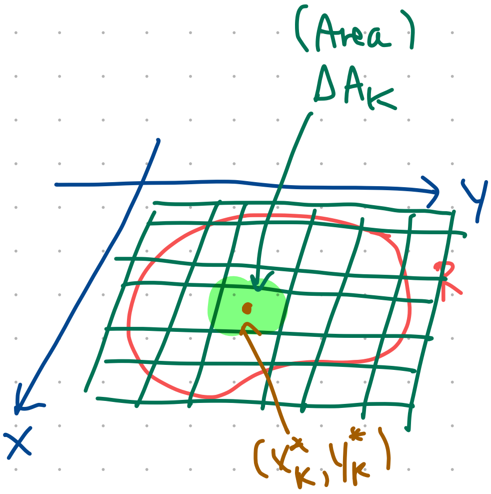
ความสูงของแท่ง ▭ $= f(x_k^*)$

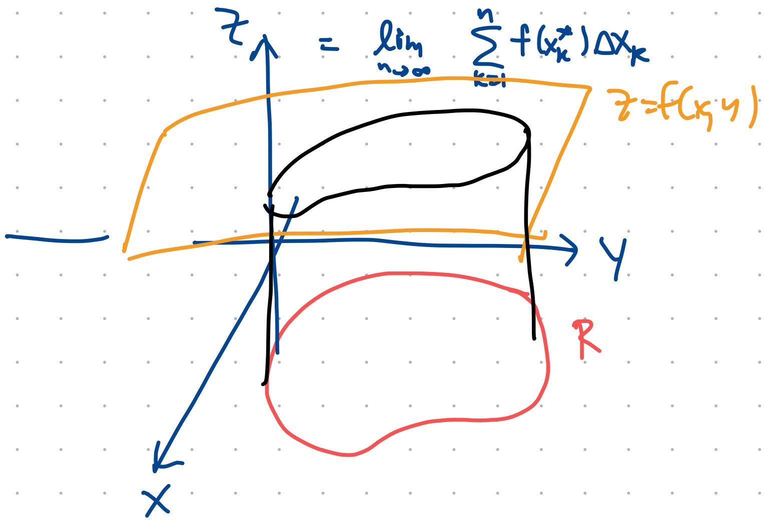


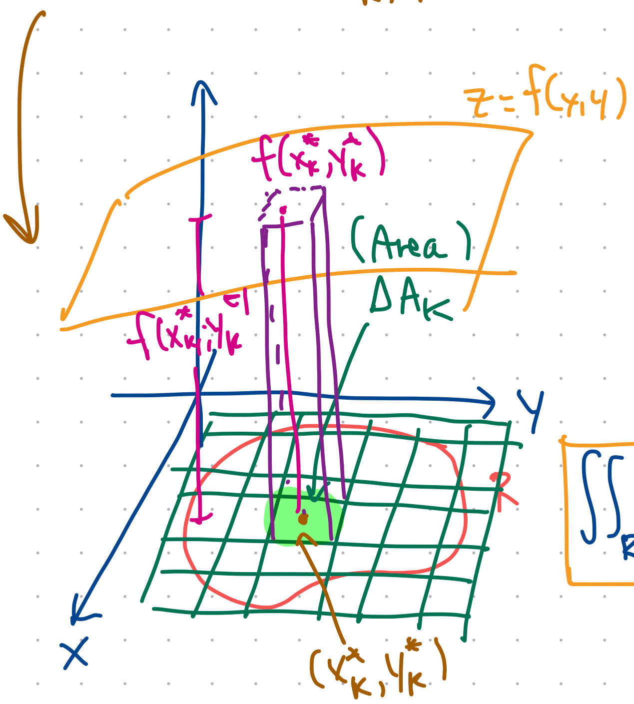
พ.ท. ▭ 1 แท่ง $= f(x_k^*)\Delta x_k$

$\therefore$ พ.ท.ใต้กราฟบน $[a,b] \Rightarrow A_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$







V 1 แท่งเล็กๆ $= f(x_k^*, y_k^*)\Delta A_k$

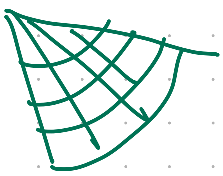
หาผลรวม n แท่ง ในบริเวณ R

$$V_n = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

(n ชิ้นในบริเวณ R)

$$\boxed{\iint_R f(x,y)\,dA = V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$$

$dA \to$ dxdy / dydx



CHAPTER 3
Multiple Integrals

3.1 Double Integrals

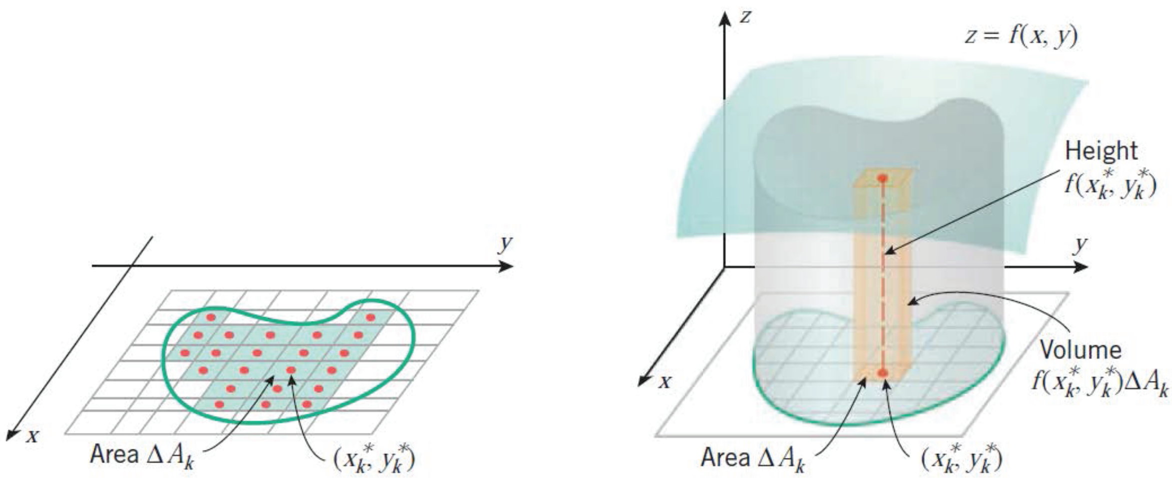
Recall that the definite integral of a function of one variable arose from the problem of nding areas under curves.

$$\int_a^b f(x)\,dx \;=\; \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k\,.$$

Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

The Volume Problem

Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1001

Definition : Volume Under a Surface

If f is a function of two variables that is *continuous and nonnegative* on a region R in the xy-plane, then the **volume of the solid enclosed between the surface** $z = f(x, y)$ **and the region** R is dened by

$$V \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k\,.$$

Here,$n \to \infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

Definition : Double Integral

If f is a function of two variables that is continuous on a region R in the xy-plane, the inetgral

$$\iint_R f(x,y)\,dA \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k\,.$$

is called the **double integral** of $f(x,y)$ over R.

Remarks. 1. If f is continuous and nonnegative on a region R, then

$$V \;=\; \iint_R f(x,y)\,dA.$$

2. If f has both positive and negative values on R, then a positive value for the double integral of f over R means that there is more volume above R than below, a negative value for the double integral means that there is more volume below R than above, and a value of zero means that the volume above R is the same as the volume below R.

Evaluating Double Integrals

Example 1. $\displaystyle\int_0^1 xy^2\,dx \;=\; \frac{x^2}{2}y^2\Big|_{x=0}^{x=1} = \frac{y^2}{2}$

$$\int_0^1 xy^2\,dy \;=\; \frac{xy^3}{3}\Big|_{y=0}^{y=1} = \frac{x}{3}$$

Iterated (or Repeated) Double Integrals a,b,c,d เป็นตัวเลข

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \;=\; \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \;=\; \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

Example 2. Evaluate

(a) $\displaystyle\int_1^3 \int_2^4 (40-2xy)\,dy\,dx$

$= \displaystyle\int_1^3 \left[\int_2^4 (40-2xy)\,dy\right] dx$

$= \displaystyle\int_1^3 \left[(40y - xy^2)\Big|_{y=2}^{y=4}\right] dx$

$= \displaystyle\int_1^3 [(160-16x)-(80-4x)]\,dx$

$= \displaystyle\int_1^3 [80-12x]\,dx$

$= 80x - \dfrac{12x^2}{2}\Big|_{x=1}^{x=3}$

$= \left(240 - \dfrac{108}{2}\right) - \left(80 - \dfrac{12}{2}\right)$

$= 112$ #

(b) $\displaystyle\int_2^4 \int_1^3 (40-2xy)\,dx\,dy$

$= \displaystyle\int_2^4 \left[\int_1^3 (40-2xy)\,dx\right] dy$

$= \displaystyle\int_2^4 \left[(40x - x^2y)\Big|_{x=1}^{x=3}\right] dy$

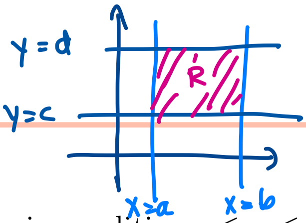
$= \displaystyle\int_2^4 [(120-9y)-(40-y)]\,dy$

$= \displaystyle\int_2^4 [80-8y]\,dy$

$= 80y - 4y^2\Big|_{y=2}^{y=4}$

$= (320-64)-(160-16)$

$= 112$ #



Fubini's Theorem

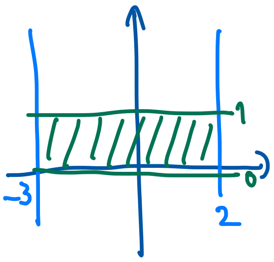
Let R be the rectangle defined by the inequalities $a \le x \le b,\ c \le y \le d$. If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example 3. Evaluate $\iint_R y^2 x\,dA$ over the rectangle $R = \{(x,y) \mid -3 \le x \le 2,\ 0 \le y \le 1\}$

Solution.

$$\therefore \iint_R y^2x\,dA = \int_{-3}^{2}\int_0^1 y^2x\,dy\,dx = \int_0^1\int_{-3}^{2} y^2x\,dx\,dy$$



พิจารณา $\int_{-3}^{2}\int_0^1 y^2x\,dydx = \int_{-3}^{2}\left[\int_0^1 y^2x\,dy\right]dx = \int_{-3}^{2} \frac{y^3}{3}x \Big|_{y=0}^{y=1} dx$

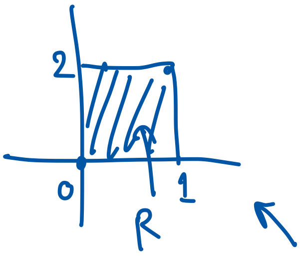
$$= \int_{-3}^{2}\left(\frac{x}{3}\right)dx = \frac{x^2}{6}\Big|_{x=-3}^{x=2} = \frac{4}{6} - \frac{9}{6} = -\frac{5}{6}$$

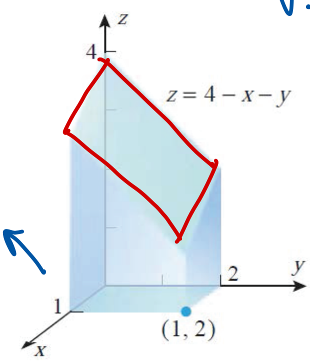
ทำแบบเดียวกัน $\int_0^1\int_{-3}^{2} y^2x\,dxdy = -\frac{5}{6}$ (Long-tom-do)

Example 4. Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0,1] \times [0,2]$ $= \{(x,y) \mid 0 \le x \le 1$ และ $0 \le y \le 2\}$

Solution.

$$V = \iint_R (4-x-y)\,dA = \int_0^2\int_0^1 (4-x-y)\,dxdy$$





$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=1} dy$$

$$= \int_0^2 \left[4 - \frac{1}{2} - y\right] dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$

$$= \frac{7}{2}y - \frac{y^2}{2}\Big|_{y=0}^{y=2} = (7-2) + (0-0)$$

$$= 5 \ \#$$

From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1005

Properties of Double Integrals

1. $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$ (c is a constant)

2. $\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

3. $\iint_R [f(x,y) - g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$

4. If $R = R_1 \cup R_2$, then $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$