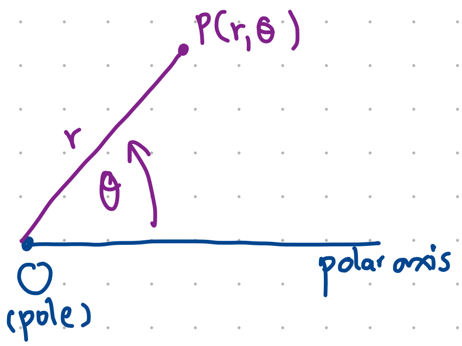
Polar Coordinate system

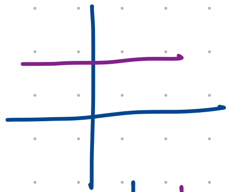


$x = r\cos\theta$
$y = r\sin\theta$ $\iff$ $x^2+y^2=r^2$
$\tan\theta = \frac{y}{x}$

① สมการเส้นตรง

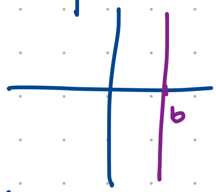
[1] ขนานแกน x (ตาม y)

$y = a \iff r\sin\theta = a$



[2] ขนานแกน y (ตาม x)

$x = b \iff r\cos\theta = b$



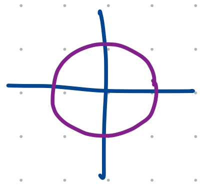
[3] ผ่านจุดกำเนิด ทำมุม θ_0 กับแกน x

$\theta = \theta_0$

② สมการวงกลม

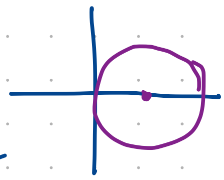
[1] สมการวงกลม รัศมี a จุดกำเนิดอยู่ที่ $(0,0)$

$x^2+y^2=a^2 \iff r = a$



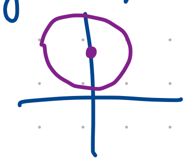
② สมการวงกลมที่ เลื่อนจุดศูนย์กลางไปตามแกน x

$r = 2a\cos\theta$

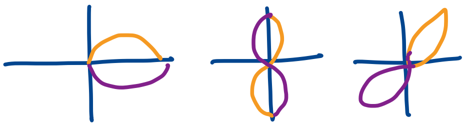


[3] สมการวงกลมที่ เลื่อนจุดศูนย์กลางไปตามแกน y

$r = 2a\sin\theta$



③ Spiral $r = f(\theta)$



Symmetry Tests

ความสมมาตร

① สมมาตรตามแกน x : แทน θ ด้วย $-\theta$ แล้วได้สมการเดิม

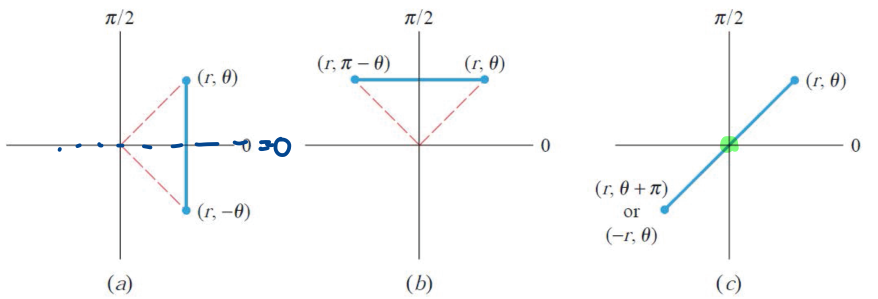
② สมมาตรตามแกน y : แทน θ ด้วย $\pi-\theta$ แล้วได้สมการเดิม

③ สมมาตรตามจุดกำเนิด: แทน θ ด้วย $\theta+\pi$ / r ด้วย $-r$

1. A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation (Figure a).

2. A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi-\theta$ in its equation produces an equivalent equation (Figure b).

3. A curve in polar coordinates is symmetric about the origin if replacing θ by $\theta+\pi$, or replacing r by r in its equation produces an equivalent equation (Figure c).



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 710

Example 6. Sketch the graph of $r = 2\cos\theta$ in polar coordinates by plotting points.

ตรวจสอบความสมมาตร: [1] สมมาตรตามแกน x : แทน θ ด้วย $-\theta$

$r = 2\cos(-\theta) = 2\cos\theta$ ได้สมการเดิม

$\therefore$ สมมาตรตามแกน x

[2] สมมาตรตามแกน y : แทน θ ด้วย $\pi-\theta$

$r = 2\cos(\pi-\theta) = -2\cos\theta$ ✗

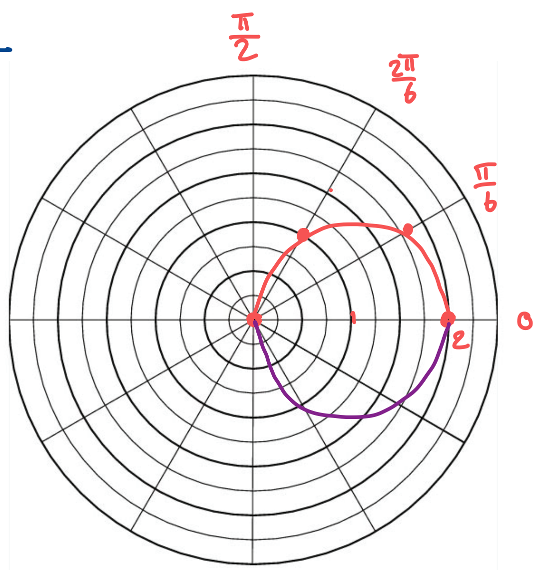
$(\cos\pi\cos\theta + \sin\pi\sin\theta)$

[3] สมมาตรตามจุดกำเนิด : แทน θ ด้วย $\pi+\theta$ หรือ $r=-r$

$\therefore -r = 2\cos\theta \Leftrightarrow r = -2\cos\theta$ ✗

$\boxed{r = 2\cos\theta}$

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{\pi}{2}$
r	2	$\sqrt{3}$	1	0



Example 7. Sketch the graph of $r = 2(1-\cos\theta)$ in polar coordinates.

Check ความสมมาตร :

$\boxed{1}$ สมมาตรรอบแกน x

$r = 2(1-\cos(-\theta))$

$r = 2(1-\cos\theta)$ ✓

สมมาตรรอบแกน x

$\boxed{2}$ สมมาตรรอบแกน y

$r = 2(1-\cos(\pi-\theta))$

$r = 2(1+\cos\theta)$ ✗

$\boxed{3}$ สมมาตรรอบจุดกำเนิด

$-r = 2(1-\cos\theta)$

$r = -2(1-\cos\theta)$ ✗

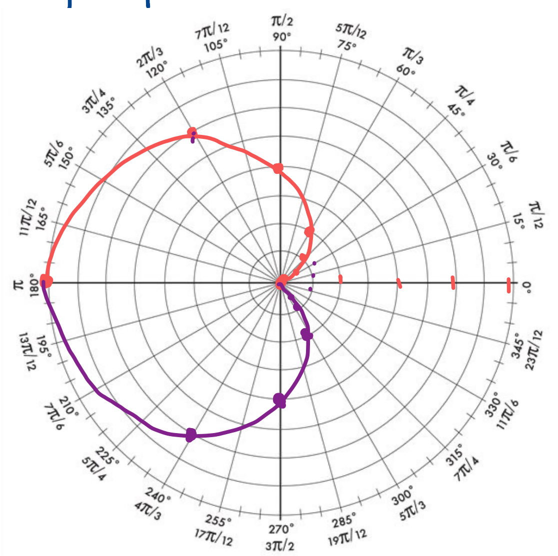
$r = 2(1-\cos\theta)$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	0	$2(1-\frac{\sqrt{3}}{2})$	$2(1-\frac{\sqrt{2}}{2})$	1	2	3	4
		≈ 0.268	≈ 0.586				

$\boxed{\text{Cardioid}}$

$r = a \pm b\sin\theta$

$r = a \pm b\cos\theta$



Example 8. Sketch the graph of $r^2 = 4\cos 2\theta$ in polar coordinates.

ทดสอบสมมาตร

$\boxed{1}$ สมมาตรรอบแกน x

$r^2 = 4\cos 2(-\theta)$

$r^2 = 4\cos 2\theta$ ✓

$\boxed{2}$ สมมาตรรอบแกน y

$r^2 = 4\cos 2(\pi-\theta)$

$r^2 = 4\cos(2\pi-2\theta)$

$\cos \overset{1}{2\pi}\cos 2\theta + \sin \overset{0}{2\pi}\sin 2\theta$

$r^2 = 4\cos 2\theta$ ✓

$\boxed{3}$ สมมาตรรอบจุด O

$(-r)^2 = 4\cos 2\theta$

$r^2 = 4\cos 2\theta$ ✓

$r^2 = 4\cos 2\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

ทด

$\theta = 0$; $r^2 = 4$

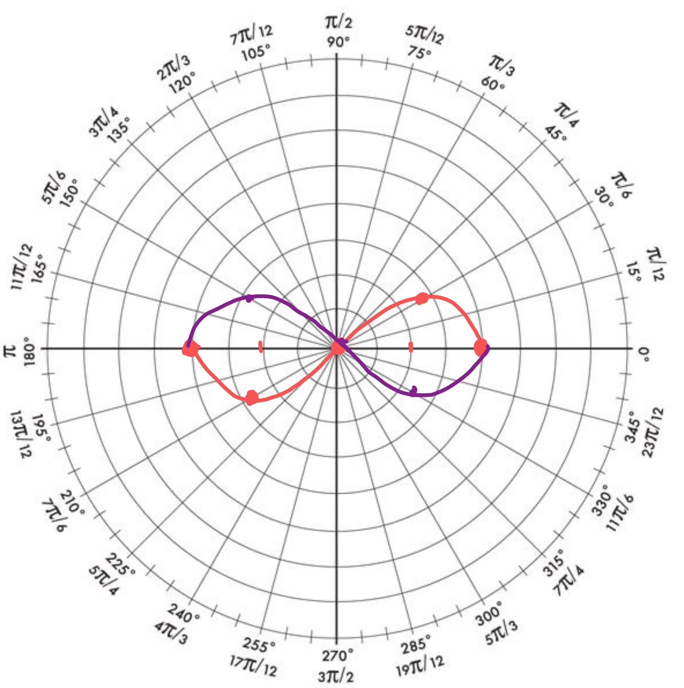
$\theta = \frac{\pi}{6}$; $r^2 = 4\cos\frac{\pi}{3}$

$r^2 = 2$

$r = \pm\sqrt{2}$

$\theta = \frac{\pi}{4}$; $r^2 = 4\cos\frac{\pi}{2}$

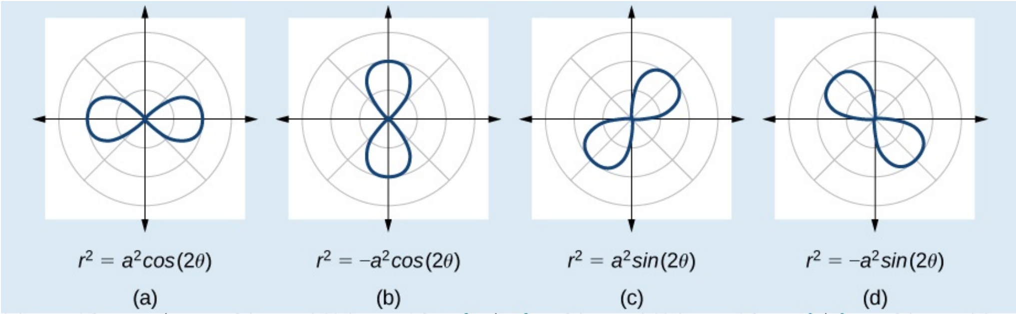
$r^2 = 0$



$\boxed{\text{lemniscate}}$

กราฟจากตัวอย่าง 4.1.13 เรียกว่า **เส้นโค้งเลมนิสเคต (Lemniscates)** ซึ่งมีสมการอยู่ในรูป

$$r^2 = \pm a^2 \cos 2\theta \qquad \text{และ} \qquad r^2 = \pm a^2 \sin 2\theta$$



รูปที่ 4.12: เส้นโค้งเลมนิสเคต

ตัวอย่าง 4.1.14. จงเขียนกราฟของเส้นโค้งที่แทนด้วยสมการเชิงขั้ว $r = 3\cos 2\theta$

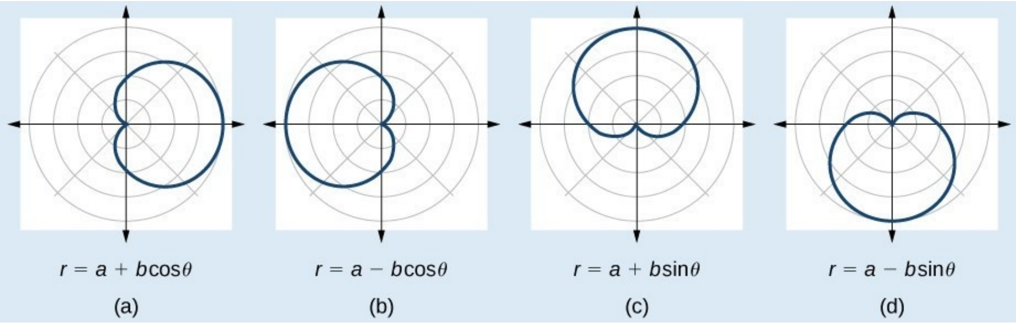
<u>วิธีทำ</u>

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r									

และสามารถเขียนกราฟได้ดังนี้

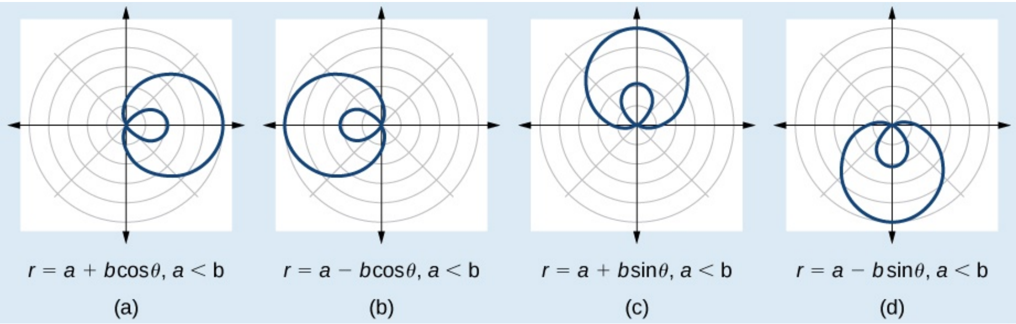
 $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$

- กรณีที่ $a = b$ เราพบว่าเส้นโค้งที่ได้จะเป็นเส้นโค้งรูปหัวใจ ดังตัวอย่าง 4.1.11 ซึ่งเรียกว่า **เส้นโค้งคาร์ดิออยด์ (cardioid)**



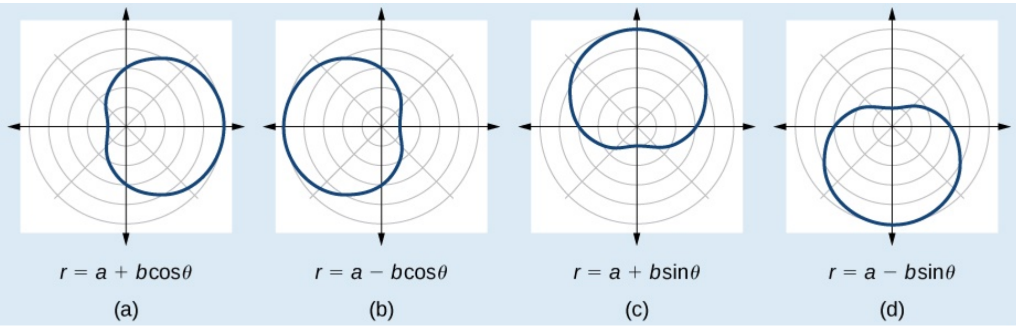
รูปที่ 4.9: เส้นโค้งคาร์ดิออยด์

- กรณีที่ $a < b$ เราพบว่าเส้นโค้งที่ได้จะเกิดเป็น inner-loop limaçon



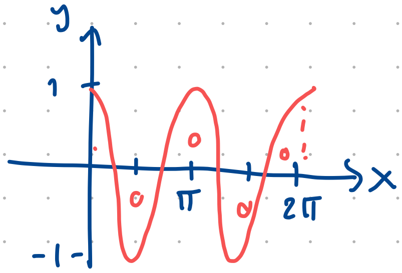
รูปที่ 4.10: เส้นโค้งลิมาคอน $a < b$

- กรณีที่ $a > b$ จะได้เส้นโค้งดังรูป Dimpled limaçon



รูปที่ 4.11: เส้นโค้งลิมาคอน $a > b$

Ex ลอง plot $r = \cos 2\theta$



1 สมมาตรตามแกน X : $r = \cos 2(-\theta)$

$\therefore r = \cos 2\theta$ ✓

2 สมมาตรตาม y : $r = \cos(2\pi - 2\theta)$

$r = \cos 2\theta$ ✓

3 สมมาตรตามจุดกำเนิด $r = \cos 2(\theta + \pi)$

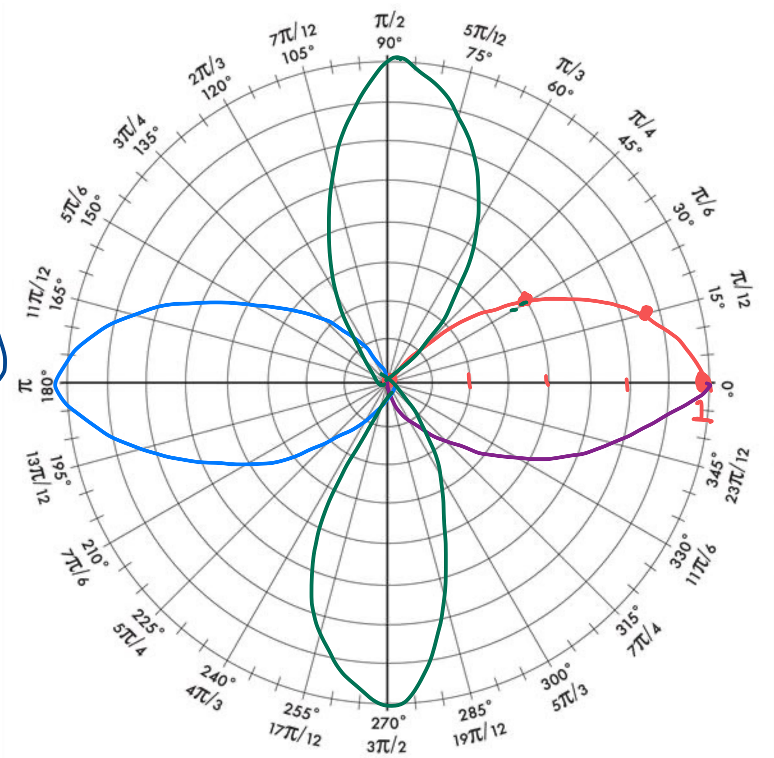
$r = \cos(2\theta + 2\pi)$

$r = \cos 2\theta$ ✓

$r = \cos 2\theta$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

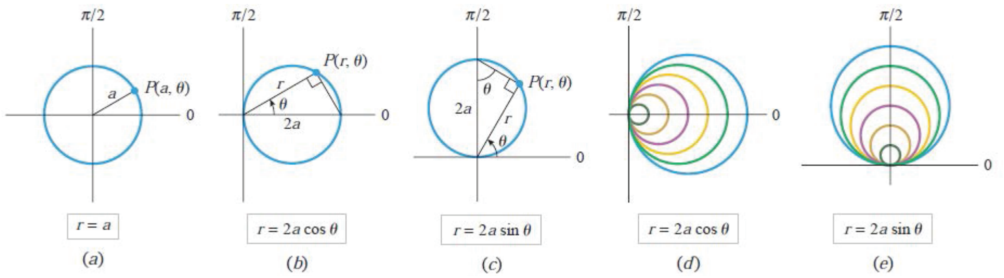
rose curves



Families of Circles

We will consider three families of circles in which a is assumed to be a positive constant:

$$r = a, \qquad r = 2a\cos\theta, \qquad r = 2a\cos\theta.$$



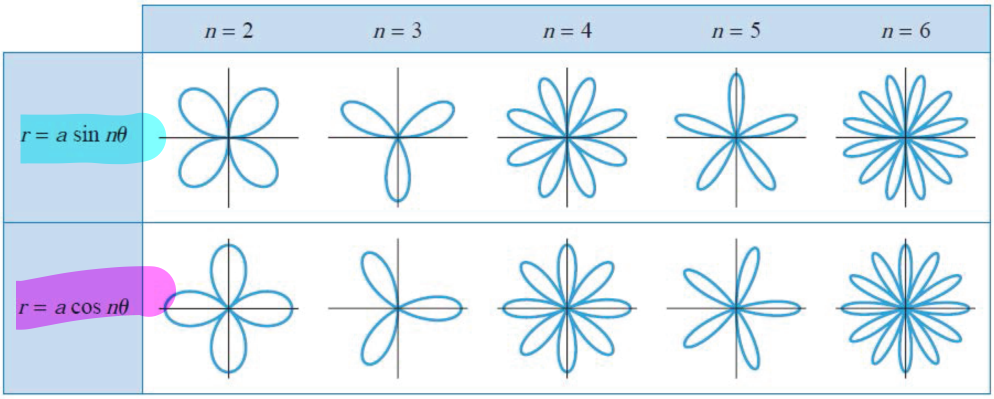
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 713

Families of Rose Curves

In polar coordinates, equations of the form

$$r = a\sin n\theta, \qquad r = a\cos n\theta$$

in which $a > 0$ and n is a positive integer represent families of flower-shaped curves called *roses*.



ถ้า n เป็นคู่
จ.น.กลีบ = 2n

ถ้า n เป็นคี่
จำนวนกลีบ = n

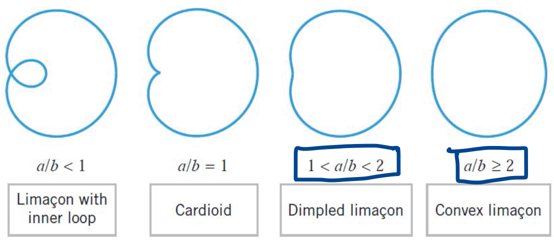
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 713

Families of Cardioids and limaons

In polar coordinates, equations of the form

$$r = a \pm b\sin\theta, \qquad r = a \pm b\cos\theta$$

in which $a > 0$ and $b > 0$ represent polar curves called *limaons* (from the Latin word limax for a snail-like creature that is commonly called a slug). There are four possible shapes for a limaon that are determined by the ratio a/b. If $a = b$ (the case $a/b = 1$), then the limaon is called a *cardioid* because of its heart-shaped appearance.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 714

13. Identify the curve by transforming the given polar equation to rectangular coordinates.

(a) $\theta = \pi/3$ (b) $\theta = -3\pi/4$ (c) $r = 3$ (d) $r = 4\cos\theta$
(e) $r = 6\sin\theta$ (f) $r - 2 = 2\cos\theta$ (g) $r = 3(1 + \sin\theta)$ (h) $r = 5 - 5\sin\theta$
(i) $r = 4 - 4\cos\theta$ (j) $r = 1 + 2\sin\theta$ (k) $r = -1 - \cos\theta)$ (l) $r = 4 + 3\cos\theta$
(m) $r = 3 - \sin\theta$ (n) $r = 3 + 4\cos\theta$ (o) $r - 5 = 3\sin\theta$ (p) $r^2 = \cos 2\theta$
(q) $r^2 = 16\sin 2\theta$ (r) $r = 4\theta$, $(\theta \geq 0)$ (s) 4θ, $(\theta \leq 0)$ (t) $r = 4\theta$
(u) $r = -2\cos 2\theta$ (v) $r = 3\sin 2\theta$ (w) $r = 9 + \sin 4\theta$ (x) $r = 2\cos 3\theta$
(y) $r = 3 + \sin\theta$ (z) $r = 2\cos\theta$

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 716-719

Double Integrals in Polar Coordinates

Simple Polar Regions

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. For example, the quarter-disk is described in rectangular coordinates by

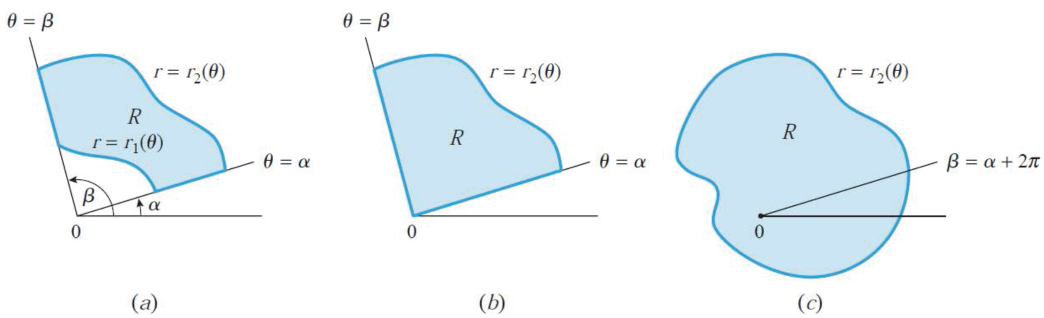
$$0 \leq y \leq \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

However, in polar coordinates the region is described more simply

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/2$$

A *simple polar region* in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves,$r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

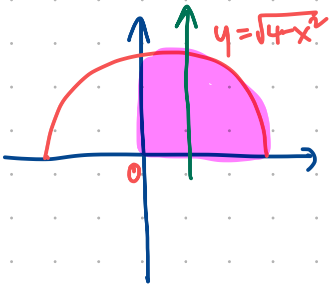
(i) $\alpha \leq \beta$ (ii) $\beta - \alpha \leq 2\pi$ (iii) $0 \leq r_1(\theta) \leq r_2(\theta)$.



Simple polar regions

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1018

ทบทวน ให้ R เป็นบริเวณที่ปิดล้อมด้วย $0 \le y \le \sqrt{4-x^2}$, $0 \le x \le 2$
จงหา พ.ท.ของบริเวณ R โดยใช้อินทิกรัลสองชั้น



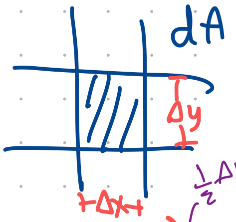
$$A = \iint_R dA$$

$$A = \int_0^2 \int_0^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_0^2 y \Big|_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

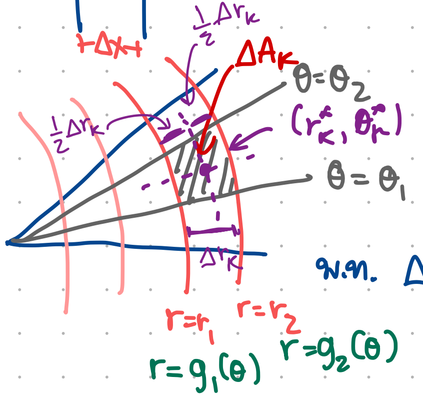
ในระบบพิกัดฉาก



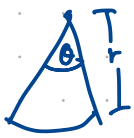
$$\Delta A = \Delta x \cdot \Delta y$$

$$dA = dx\,dy$$

ในพิกัดเชิงขั้ว



พ.ท. ของรูป Δ ฐานโค้ง
(θ เป็นเรเดียน)



$$A = \frac{1}{2} r^2 \theta$$

พ.ท. $\Delta A_k = \frac{1}{2} r_2^2(\theta_2-\theta_1) - \frac{1}{2} r_1^2(\theta_2-\theta_1)$

$$= \frac{1}{2}(r_2^2 - r_1^2)(\theta_2-\theta_1)$$

$$= \underbrace{\left(\frac{r_1+r_2}{2}\right)}_{r_k^*}\underbrace{(r_2-r_1)}_{\Delta r}\underbrace{(\theta_2-\theta_1)}_{\Delta\theta}$$

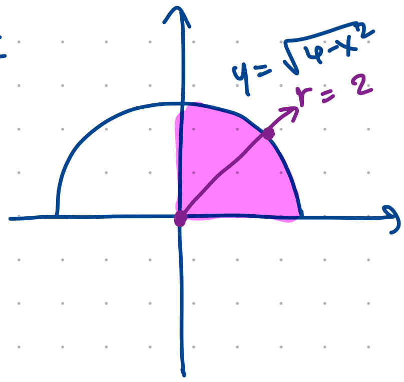
$$\Delta A_k = r_k^* \Delta r \cdot \Delta\theta$$

$$dA = r\,dr\,d\theta$$

$$V = \iint_R f(r,\theta)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\, r^* \Delta r_k\, \Delta\theta_k$$

$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

Ex



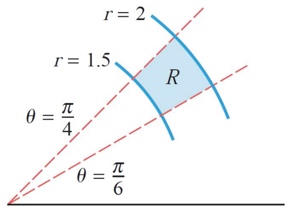
$$A = \iint_R dA = \int_0^{\frac{\pi}{2}} \int_0^2 r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2} \Big|_{r=0}^{r=2} d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\theta \Big|_{\theta=0}^{\pi/2} = \pi \;\#$$

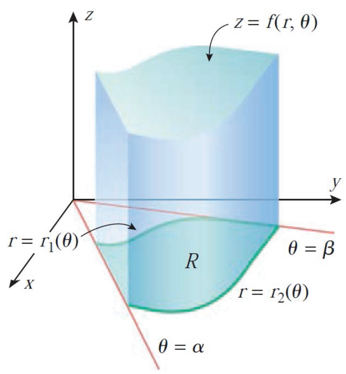
A *polar rectangle* is a simple polar region for which the bounding polar curves are circular arcs. For example, in below figure shows the polar rectangle R given by

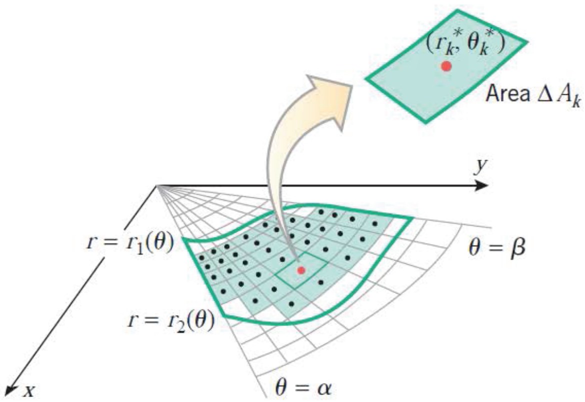
$$1.5 \leq r \leq 2, \quad \pi/6 \leq \theta \leq \pi/4.$$

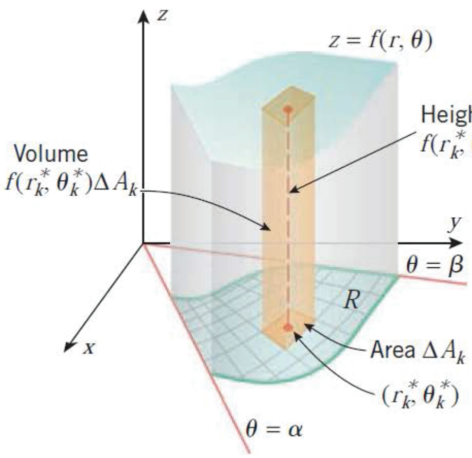


The Volume Problem in Polar Coordinates

Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$.







From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1019-1020

The volume of the solid is

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k.$$

If $f(r, \theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

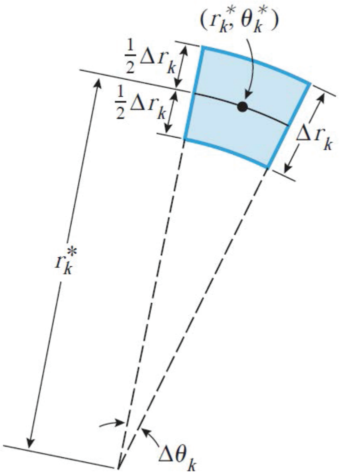
represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). These sums are called *polar Riemann sums*, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r,\theta)\, dA \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

which is called the *polar double integral* of $f(r,\theta)$ over R. If $f(r,\theta)$ is continuous and nonnegative on R, then the volume formula can be expressed as

$$V \;=\; \iint_R f(r,\theta)\, dA$$

Evaluating Polar Double Integrals



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1020

Treating the area ΔA_k of this polar rectangle as the difference in area of two sectors, we obtain

$$\Delta A_k \;=\; \frac{1}{2}(r_k^* + \frac{1}{2}\Delta r_k)^2 \Delta\theta_k - \frac{1}{2}(r_k^* - \frac{1}{2}\Delta r_k)^2 \Delta\theta_k$$

which simplifies to

$$\Delta A_k \;=\; r_k^* \Delta r_k \Delta\theta_k\,.$$

Thus

$$V \;=\; \iint_R f(r,\theta)\, dA \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

which suggests that the volume V can be expressed as the iterated integral

$$V \;=\; \iint_R f(r,\theta)\, dA \;=\; \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta\,.$$

Theorem. If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ and if $f(r,\theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA \;=\; \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta\,.$$