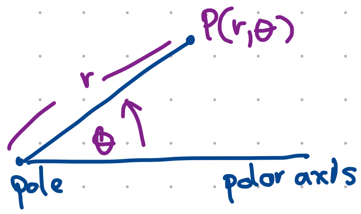
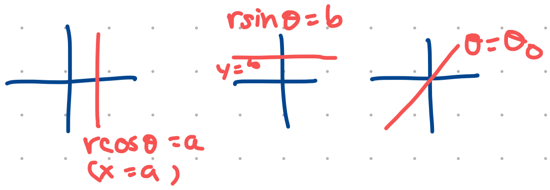


① เส้นตรง



② วงกลม

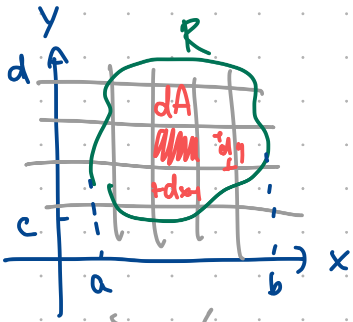
③ Cardioid + limaçon

④ Lemniscate

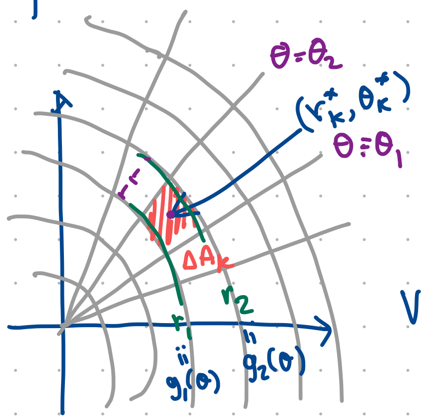
⑤ rose curve

⑥ spiral

การทดสอบสมมาตร
- สมมาตรแกน x (θ ด้วย $-\theta$)
- สมมาตรแกน y (θ ด้วย $\pi-\theta$)
- สมมาตรจุดกำเนิด (θ ด้วย $\pi+\theta$ หรือ r ด้วย $-r$)



$$\iint_R f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



คาบที่แล้วทราบว่า

$$\Delta A_k = \underbrace{\left(\frac{r_1+r_2}{2}\right)}_{r_k^*}\underbrace{(r_2-r_1)}_{\Delta r}\underbrace{(\theta_2-\theta_1)}_{\Delta\theta}$$

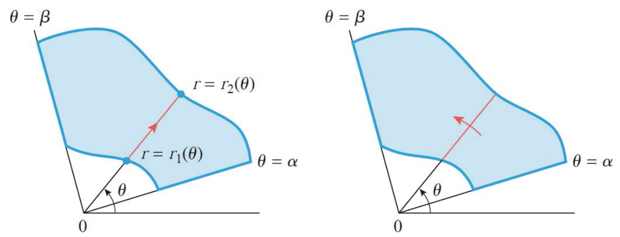
$$V = \iint_R f(r,\theta)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(r_k^*, \theta_k^*)\, r_k^* \Delta r_k \Delta\theta_k$$

$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

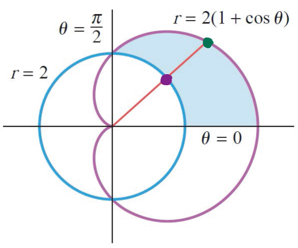
Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a xed angle θ. This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration.

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$. This determines the θ-limits of integration.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1021

Example 9. Evaluate $\iint_R \sin\theta\, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.



From:Calculus Early Transcendentals, 10th edition,Howard Anton,Irl C. Beven, Stephen Deavis, page 1022

$$\iint_R \sin\theta\, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} (\sin\theta)\, r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^2}{2}\sin\theta \Big|_{r=2}^{r=2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2}\left(\frac{4(1+\cos\theta)^2}{2}\sin\theta - \frac{4}{2}\sin\theta\right)d\theta$$

$$= 2\int_0^{\pi/2}(1+\cos\theta)^2\sin\theta\,d\theta - 2\int_0^{\pi/2}\sin\theta\,d\theta$$

$$-2\int_0^{\pi/2}\sin\theta\,d\theta = 2\cos\theta\Big|_{\theta=0}^{\pi/2} = -2$$

ให้ $u = 1+\cos\theta$; $du = -\sin\theta\,d\theta$

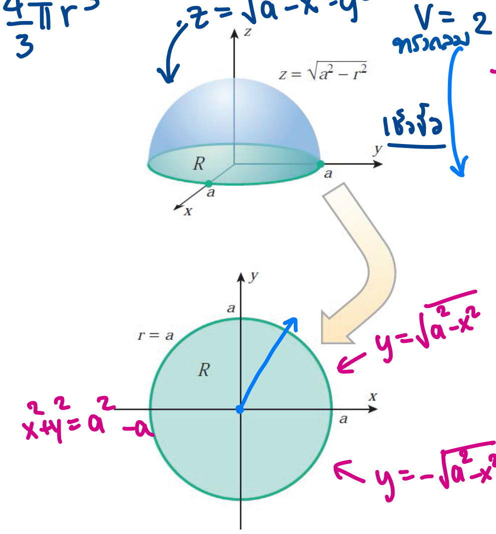
$$\therefore \int(1+\cos\theta)^2\sin\theta\,d\theta = -\int u^2\,du = -\frac{u^3}{3} = -\frac{(1+\cos\theta)^3}{3}$$

$$\therefore 2\int_0^{\pi/2}(1+\cos\theta)^2\sin\theta\,d\theta = -2\,\frac{(1+\cos\theta)^3}{3}\Big|_{\theta=0}^{\pi/2} = -\frac{2}{3}[1-8] = \frac{14}{3}$$

$$= \frac{14}{3} - 2 = \frac{8}{3} \quad \#$$

Example 10. The sphere of radius a centered at the origin is expressed in rectangu-lar coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to nd the volume of the sphere.

$V = \frac{4}{3}\pi r^3$



$$V_{\text{ทรงกลม}} = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx \quad (\text{พิกัดฉาก})$$

เชิงขั้ว

$$z = \sqrt{a^2-x^2-y^2} = \sqrt{a^2-(x^2+y^2)} = \sqrt{a^2-r^2}$$

$$V = \int_0^{2\pi}\int_0^{a} 2\sqrt{a^2-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(-\frac{2}{3}(a^2-r^2)^{3/2}\right)\Big|_{r=0}^{r=a} d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(0) + \frac{2}{3}(a^2)^{\frac{3}{2}}\right]d\theta = \frac{2}{3}\int_0^{2\pi} a^3\,d\theta = \frac{2}{3}a^3\,[\theta]_{\theta=0}^{2\pi}$$

$$= \frac{4\pi}{3}a^3 \quad \#$$

หา

ให้ $u = a^2 - r^2$; $du = -2r\,dr$

$$\therefore \int 2r\sqrt{a^2-r^2}\,dr = \int -\sqrt{u}\,du = -\frac{2u^{3/2}}{3} + C = -\frac{2}{3}(a^2-r^2)^{3/2} + C$$

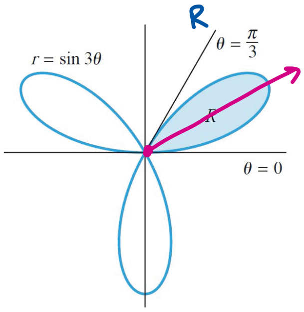
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1022

Finding Areas Using Polar Double Integrals

Recall that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA\,.$$

Example 11. Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.



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$$A = 3\iint_R dA = 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta = 3\int_0^{\pi/3} \frac{r^2}{2}\Big|_{r=0}^{r=\sin 3\theta} d\theta$$

$$= 3\int_0^{\pi/3} \frac{\sin^2 3\theta}{2}\, d\theta \qquad \boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$= \frac{3}{2}\int_0^{\pi/3} \frac{(1-\cos 6\theta)}{2}\, d\theta$$

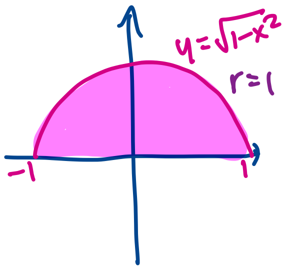
$$= \frac{3}{4}\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta = \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_{\theta=0}^{\pi/3}$$

$$= \frac{\cancel{3}}{4}\left[\left(\frac{\pi}{\cancel{3}} - 0\right) - (0-0)\right] = \frac{\pi}{4} \quad \#$$

Converting Double Integrals from Rectangle to Polar Coordinates Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta,\ y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta)\; r\,dr\,d\theta\,.$$

Example 12. Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$



เดิม $f(x,y) = (x^2+y^2)^{3/2}$

$\Updownarrow$

$f(r,\theta) = (r^2)^{3/2} = r^3$

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int_0^{\pi}\int_0^{1} (r^3)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{1} r^4\,dr\,d\theta = \int_0^{\pi} \frac{r^5}{5}\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi} \frac{1}{5}\,d\theta = \frac{\theta}{5}\Big|_{\theta=0}^{\theta=\pi} = \frac{\pi}{5} \quad \#$$

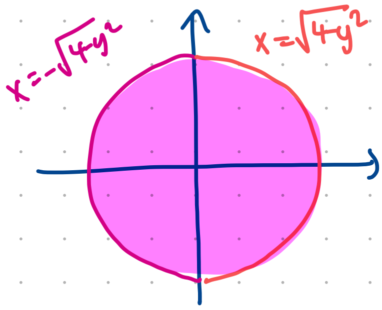
จงแปลงอินทิกรัลต่อไปนี้ในรูปพิกัดเชิงขั้ว

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$$

จงแปลงอินทิกรัลต่อไปนี้ในรูปพิกัดเชิงขั้ว

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$$

$x = -\sqrt{4-y^2}$



① $f(x,y) = e^{-(x^2+y^2)}$

$\Updownarrow$

$f(r,\theta) = e^{-r^2}$

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy = \int_{0}^{2\pi}\int_{0}^{2} e^{-r^2}\,r\,dr\,d\theta$$

จงแปลงอินทิกรัลต่อไปนี้ในรูปพิกัดเชิงขั้ว

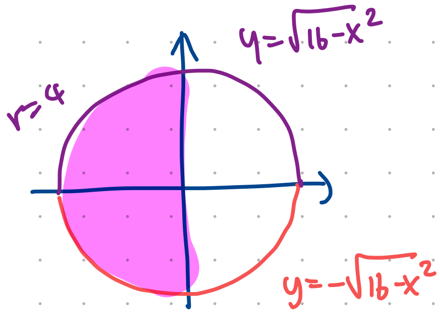
$$\int_{-4}^{0}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\,dy\,dx$$

y y

$f(x,y) = 3x$

$\Updownarrow$

$f(r,\theta) = 3r\cos\theta$



$$\int_{-4}^{0}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\,dy\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{4} (3r\cos\theta)\,r\,dr\,d\theta$$