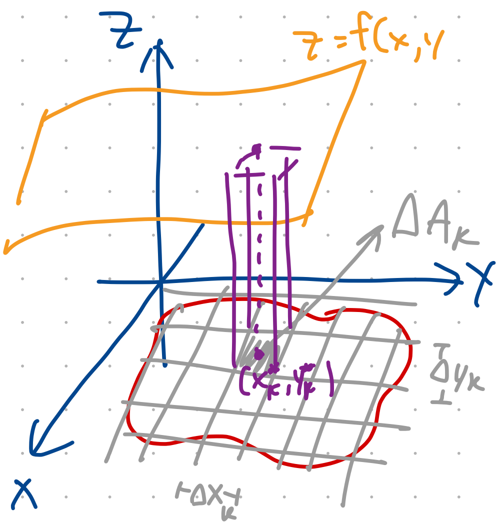
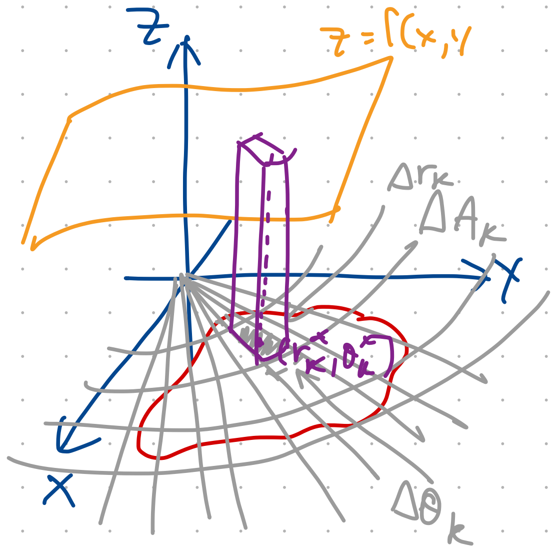
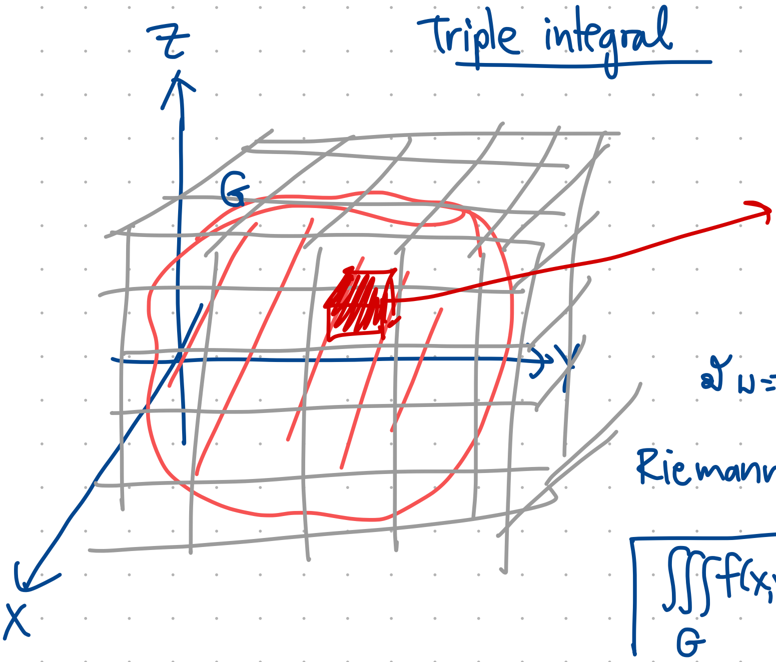
Double integral



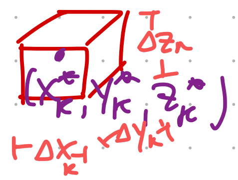


$\Downarrow$

Triple integral



Volume $= \Delta V_k = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$



$= \underbrace{\Delta y_k \, \Delta x_k \, \Delta z_k}_{\text{สลับได้ 6 แบบ}}$

ให้ $w = f(x,y,z)$

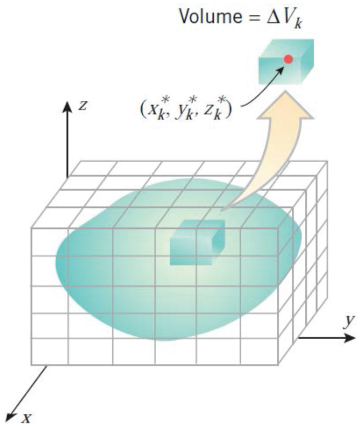
Riemann sum $= \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta V_k$

$$\boxed{\iiint_G f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta V_k}$$

3.4 Triple Integrals in Rectangular Coordinates

Definition of a Triple Integral

To define the triple integral of $f(x, y, z)$ over G, we rst divide the box into n subboxes by planes parallel to the coordinate planes. We then discard those subboxes that contain any points outside of G and choose an arbitrary point in each of the remaining subboxes. We denote the volume of the kth remaining subbox by ΔV_k and the point selected in the kth subbox by (x_k^*, y_k^*, z_k^*).



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1039

Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

for each subbox, then add the products for all of the subboxes to obtain the *Riemann sum*

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k\,.$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each subbox approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x, y, z)\, dV \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k\,.$$

Properties of Triple Integrals

Triple integrals enjoy many properties of single and double integrals:

$$\iiint_G cf(x, y, z)\, dV \;=\; c\iiint_G f(x, y, z)\, dV \qquad (c \text{ a constant})$$

$$\iiint_G \big[f(x, y, z) + g(x, y, z]\big)\, dV \;=\; \iiint_G f(x, y, z)\, dV + \iiint_G g(x, y, z)\, dV$$

$$\iiint_G \big[f(x, y, z) - g(x, y, z]\big)\, dV \;=\; \iiint_G f(x, y, z)\, dV - \iiint_G g(x, y, z)\, dV$$

Moreover, if the region G is subdivided into two subregions G_1 and G_2 , then

$$\iiint_G f(x, y, z)\, dV \;=\; \iiint_{G_1} f(x, y, z)\, dV + \iiint_{G_2} f(x, y, z)\, dV$$

Double integral ① ②

Evaluating Triple Integrals over Rectangular Boxes

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations.

Fubini's Theorem Let G be the rectangular box defined by the inequalities

$$a \le x \le b\,, \quad c \le y \le d\,, \quad k \le z \le l\,.$$

If f is continuous on the region G, then

$$\iiint_G f(x,y,z)\,dV = \left(\int_a^b \left(\int_c^d \left(\int_k^l f(x,y,z)\,dz\right) dy\right) dx\right)$$

dzdydx
dzdxdy
dxdzdy
dxdydz
dydxdz
dydzdx

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration:

$$dydzdx,\ dzdxdy,\ dxdzdy,\ dzdydx,\ dydxdz$$

Example 1. Evaluate the triple integral $\iiint_G 12xy^2z^3\,dV$ over the rectangular box G dened by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

$$\iiint_G 12xy^2z^3\,dV = \int_{-1}^{2}\int_{0}^{2}\int_{0}^{3}(12xy^2z^3)\,dy\,dz\,dx$$

$$= \int_{-1}^{2}\int_{0}^{2} \cancel{12}^{4} x z^3 \cdot \frac{y^3}{\cancel{3}} \Big|_{y=0}^{y=3} dz\,dx$$

$$= 4\int_{-1}^{2}\int_{0}^{2} 27\cdot x z^3\,dz\,dx$$

$$= 4\cdot 27\int_{-1}^{2} x\cdot\frac{z^4}{4}\Big|_{z=0}^{z=2} dx$$

$$= 4\cdot 27\int_{-1}^{2} x\cdot\frac{\cancel{16}}{\cancel{4}}\,dx$$

$$= 27\cdot 16\int_{-1}^{2} x\,dx$$

$$= 27\cdot 16\left[\frac{x^2}{2}\right]_{x=-1}^{2}$$

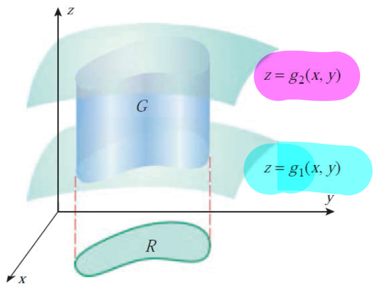
$$= 27\cdot 16\left[\frac{4}{2}-\frac{1}{2}\right] = 27\cdot\cancel{16}^{8}\cdot\frac{3}{\cancel{2}}$$

$$= 648$$

Evaluating Triple Integrals over More general Regions

Theorem. Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G, then

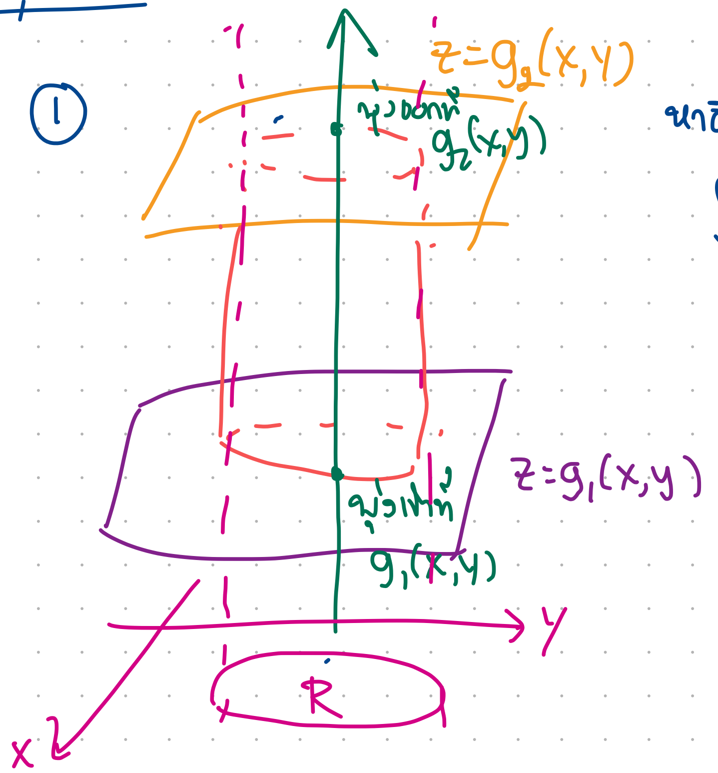
$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA\,.$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1041

สรุปขั้นตอน

①

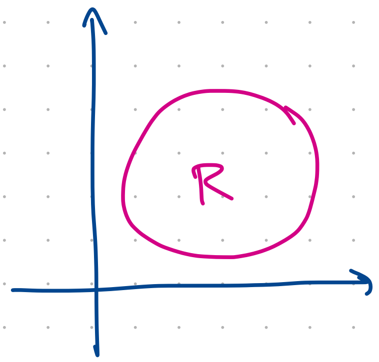


หาอินทิกรัลตามชั้น

$$\iiint_G f(x,y,z)\,dV = \iint \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right] dA$$

② อินทิเกรตตัวนอก , ดู projection ของ G ลงบนระนาบ xy

อินทิเกรตขอบเขตบริเวณ R เหมือนการอินทิเกรตสองชั้น



→ Double Integral : Set $f(x,y) = 1$; $A = \iint_R dA$

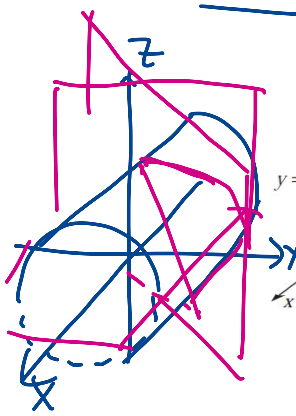
Triple integral : Set $f(x,y,z) = 1$; $V = \iiint_G dV$

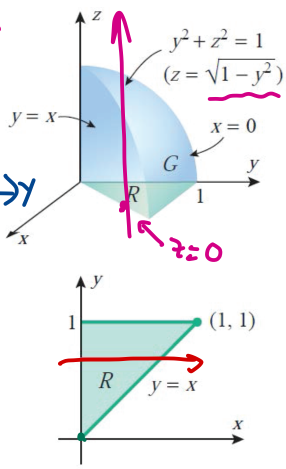
Determining Limits of Integration: Simple xy-Solid

Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R

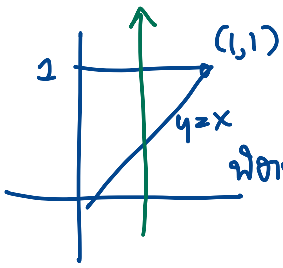
Example 2. Let G be the wedge (ลิ่ม) in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate $\iiint_G z\, dV$





$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

$$= \int_0^1 \int_x^1 \int_0^{\sqrt{1-y^2}} z\,dz\,dy\,dx$$



พิจารณา $\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{z^2}{2}\Big|_{z=0}^{z=\sqrt{1-y^2}} dx\,dy$

$$= \int_0^1 \int_0^y \left(\frac{1-y^2}{2}\right) dx\,dy$$

$$= \int_0^1 \left(\frac{1-y^2}{2}\right) y \,dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{8}$$

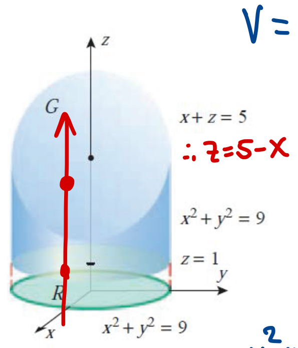
\#

From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1042

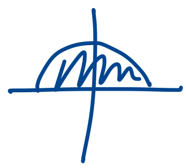
Volume calculated as a triple Integral : volume of $G = \iiint_G dV$

Example 3. Use a triple integral to nd the volume of the solid within the cylinder $x^2+y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

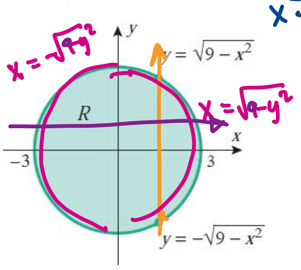
$$V = \iiint_G dV = \iint_R \left[\int_1^{5-x} 1\,dz\right] dA$$



$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-x} 1\,dz\,dy\,dx$$



$$= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_1^{5-x} dz\,dx\,dy$$



$x^2 + y^2 = 9$

$x^2 = 9 - y^2$

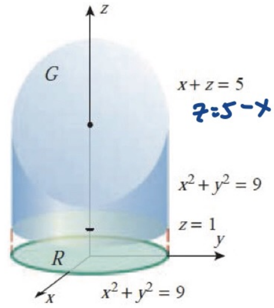
$x = \pm\sqrt{9-y^2}$

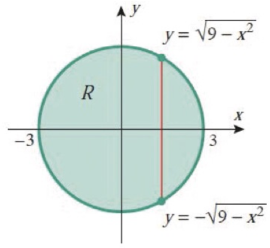
36π

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1043

เฉลยข้อเมื่อกี้ :)

Example 3. Use a triple integral to nd the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$.





$$V = \iiint_G dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-x} dz\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{z=1}^{z=5-x} dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-x-1)\,dy\,dx$$

$$= \int_{-3}^{3} (4-x)y\Big|_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3}\left[(4-x)\sqrt{9-x^2} + (4-x)\sqrt{9-x^2}\right] dx$$

$$= \int_{-3}^{3} 2\sqrt{9-x^2}\,(4-x)\,dx = \int_{-3}^{3} 8\sqrt{9-x^2}\,dx - \boxed{\cancel{\int_{-3}^{3} 2x\sqrt{9-x^2}\,dx}}\ 0$$

$$= 8\cdot\frac{9}{2}\pi = 36\pi \quad \#$$

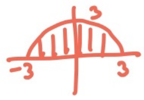
ให้ $u = 9-x^2$; $du = -2x\,dx$

$\therefore \int 2x\sqrt{9-x^2}\,dx = -\int \sqrt{u}\,du$

$= -\frac{2u^{3/2}}{3} + C$

$= -\frac{2}{3}(9-x^2)^{3/2} + C$

if $x = 3, -3$
Then it cancel each other!



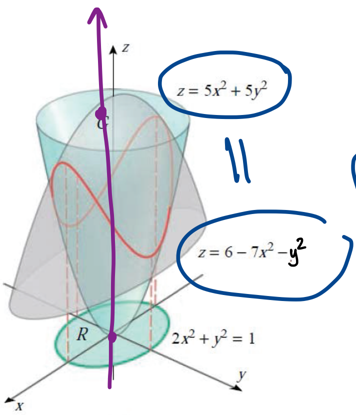
Area $= \frac{\pi(3)^2}{2} = \frac{9}{2}\pi$

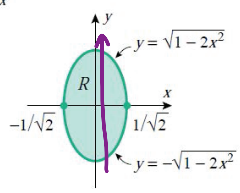
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1043

Set $f(x,y,z) = 1$

Example 4. Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$





$$V = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz \right] dA$$

หา projection ลงบนระนาบ xy

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$
$$12x^2 + 6y^2 = 6$$
$$2x^2 + y^2 = 1$$
$$\frac{x^2}{\frac{1}{2}} + \frac{y^2}{1} = 1$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\,dy\,dx$$

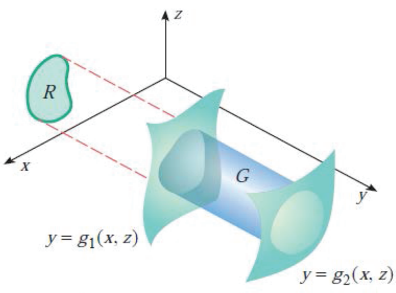
From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1043

Integration in Other Orders

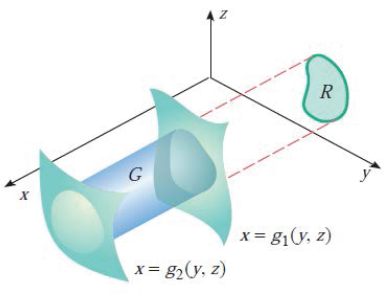
For integrating over a simple xy-solid, the z-integration was performed first. However, there are situations in which it is preferable to integrate in a different order. For a simple xz-solid it is usually best to integrate with respect to y first, and for a simple yz-solid it is usually best to integrate with respect to x first:

$$\underset{\substack{G \\ \text{simple } xz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy \right] dA$$

$$\underset{\substack{G \\ \text{simple } xz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \iint_R \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx \right] dA.$$



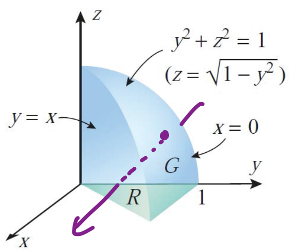
A simple xz-solid



A simple yz-solid

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1043

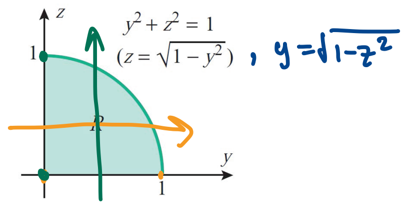
Example 5. Evaluate this integral $\iiint_G z\,dV$ in Example 2 by integrating first with respect to x.



$$\iiint_G z\,dV = \iint_R \left[\int_0^y z\,dx \right] dA$$

$$= \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^y z\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\,dx\,dz\,dy$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1045

EXERCISES 3.4

1. The iterated integral $\int_1^5 \int_2^4 \int_3^6 f(x,y,z)\,dx\,dz\,dy$ integrates f over the rectangular box defined by

$$\ldots\ldots\ldots \le x \le \ldots\ldots\ldots, \quad \ldots\ldots\ldots \le y \le \ldots\ldots\ldots, \quad \ldots\ldots\ldots \le z \le \ldots\ldots\ldots$$

2. Let G be the solid in the first octant bounded below by the surface $z = y + x^2$ and bounded above by the plane $z = 4$. Supply the missing limits of integration.

$$(a)\ \iiint_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_{y+x^2}^4 f(x,y,z)\,dz\,dx\,dy$$

$$(b)\ \iiint_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_{y+x^2}^4 f(x,y,z)\,dz\,dy\,dx$$

$$(c)\ \iiint_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_\square^\square f(x,y,z)\,dy\,dz\,dx$$

3-10 Evaluate the iterated integral.

3. $\int_{-1}^1 \int_0^2 \int_0^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$

4. $\int_{1/3}^{1/2} \int_0^\pi \int_0^1 zx \sin xy\,dz\,dy\,dx$

5. $\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yz\,dx\,dz\,dy$