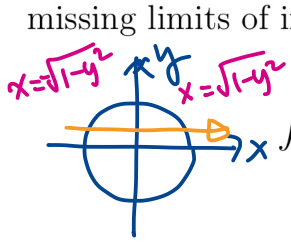
19. Let G be the solid enclosed by the surfaces in the accompanying figure. Fill in the missing limits of integration.



$$\iiint_G f(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dy\,dx$$

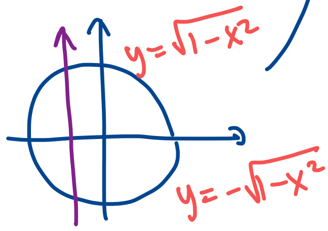
$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dx\,dy$$

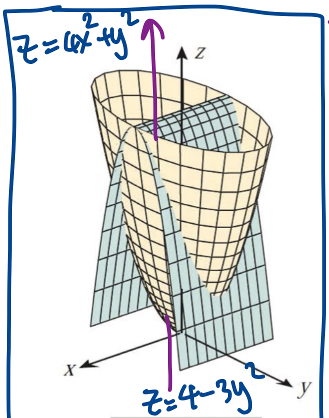
หา projection : ดูรอยตัด

$4x^2+y^2 = 4-3y^2$

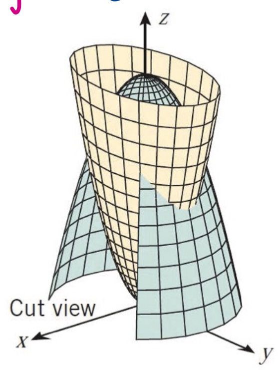
$4x^2+4y^2 = 4$

$x^2+y^2 = 1$



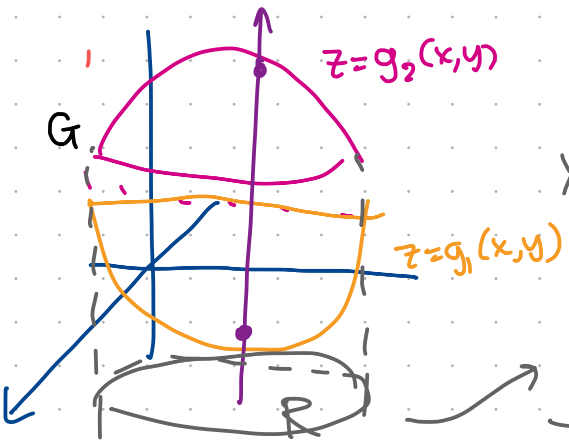


$z = 4x^2 + y^2$
$z = 4 - 3y^2$



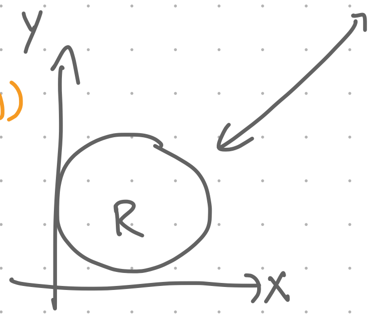
$z = 3x^2 + y^2$
$z = 8 - x^2 - y^2$

Triple integral



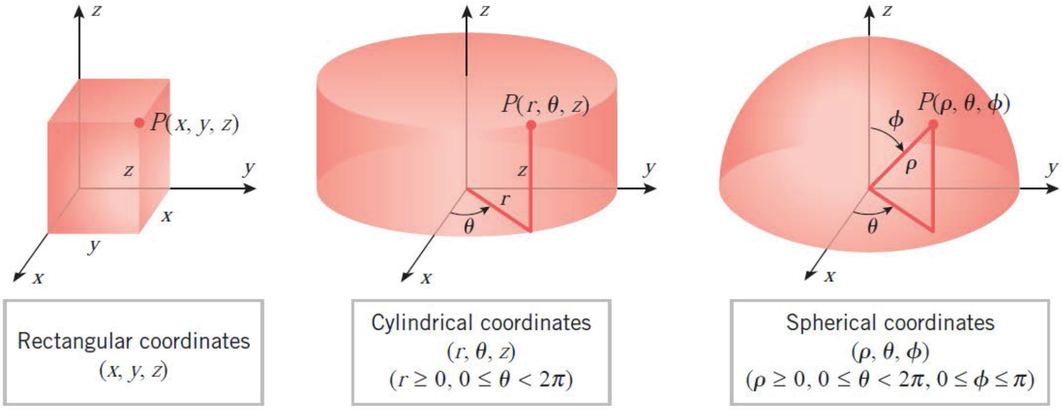
R : projection ของ กราฟของ G

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA$$



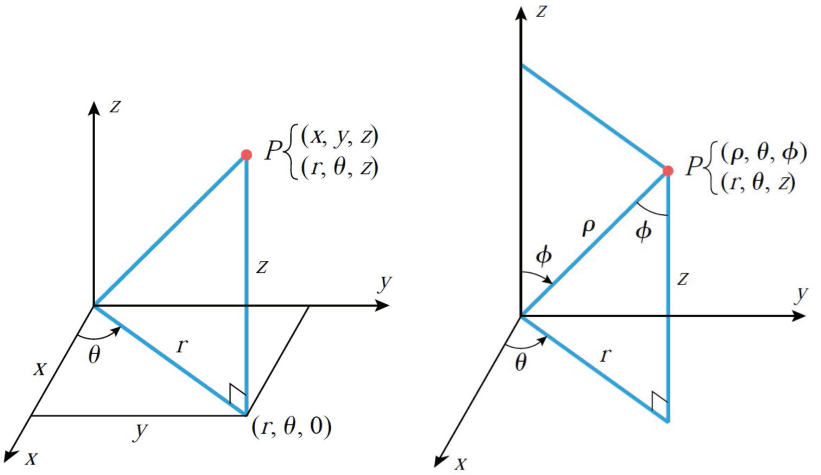
3.5 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical and Spherical Coordinate Systems



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 832

Converting Coordinates



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 834

Cylindrical to Rectangular

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Rectangular to Cylindrical

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

Spherical to Rectangular

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

Rectangular to Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

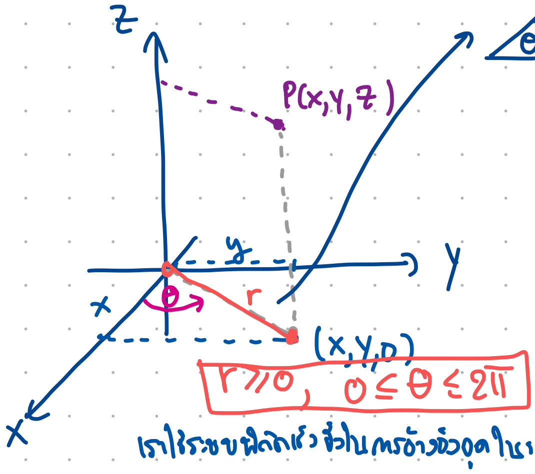
Spherical to Cylindrical

$$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$$

Cylindrical to Spherical

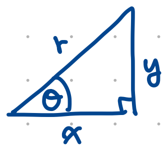
$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \tan\phi = \frac{r}{z}$$

Cylindrical Coordinate



$\boxed{r \geq 0, \quad 0 \leq \theta \leq 2\pi}$

เราใช้ระบบพิกัดเชิงขั้วในการอ้างอิงจุดในระนาบ xy



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

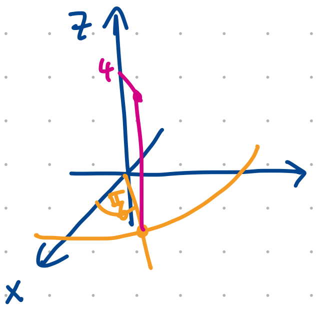
$\Updownarrow$

$$\boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}}$$

$r \geq 0$

$0 \leq \theta \leq 2\pi$

Ex $(5, \frac{\pi}{6}, 4)$ แปลงเป็นระบบพิกัดฉาก



$x = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$

$y = 5\sin\frac{\pi}{6} = \frac{5}{2}$

$z = 4$

$\therefore (5, \frac{\pi}{6}, 4) = (\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4)$

$(3, -3, -7)$ ให้เป็นพิกัดทรงกระบอก $(r \geq 0, \ 0 \leq \theta \leq 2\pi)$

$x^2+y^2 = r^2$

$3^2+(-3)^2 = r^2$

$r^2 = 18$

$r = \pm\sqrt{18}$

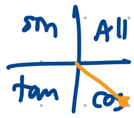
$(r = \sqrt{18})$

$\tan\theta = \frac{y}{x}$

$\tan\theta = -\frac{3}{3}$

$\tan\theta = -1$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$



$\therefore (3,-3,-7)$ คือ $(\sqrt{18}, \frac{7\pi}{4}, -7)$

Constant surface

$\boxed{1}$ Ex $\theta = \frac{\pi}{4}$ ในระบบพิกัดทรงกระบอก

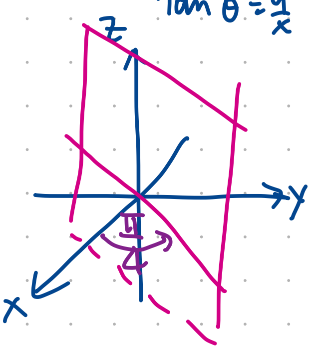
$\downarrow$

$\tan\theta = \frac{y}{x} \Leftrightarrow \tan\frac{\pi}{4} = \frac{y}{x}$

$\therefore y = x$

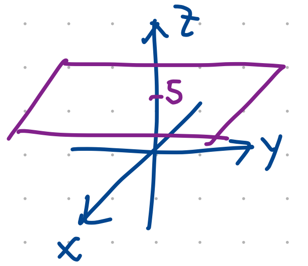
$r^2 = x^2+y^2$

$\tan\theta = \frac{y}{x}$



$\Rightarrow$ $\theta = \theta_0 \Rightarrow$ เป็นแผ่นระนาบ ที่ทำมุม θ_0 กับแกน x และตั้งฉากกับระนาบ xy

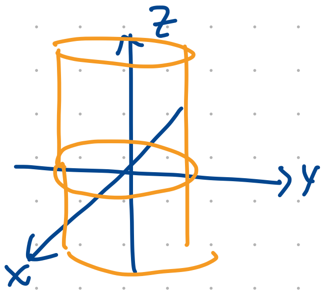
$\boxed{3}$ $z = 5$ แผ่นระนาบที่



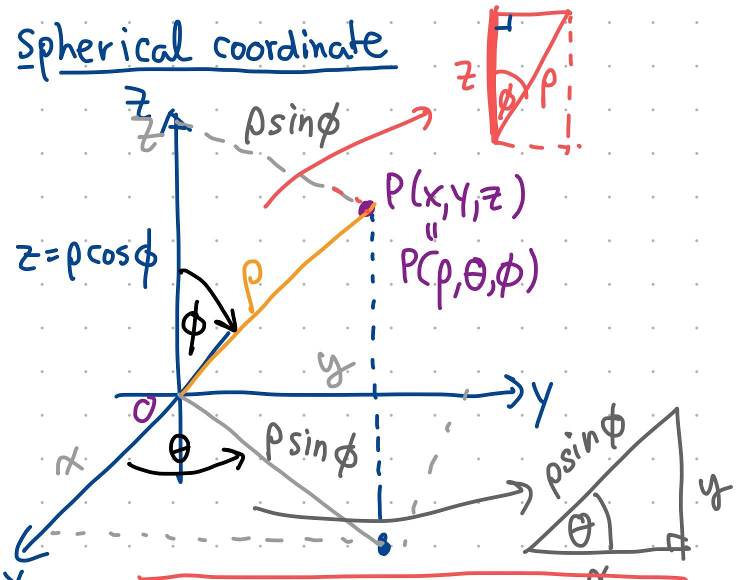
$\boxed{2}$ Ex $r = 5$ ในระบบพิกัดทรงกระบอก

$x^2+y^2 = r^2 \Leftrightarrow x^2+y^2 = 5^2$

$r = r_0 \Rightarrow$ สมการที่ได้เป็นทรงกระบอกกลม ตามแกน z, รัศมี r_0



Spherical coordinate



$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$

$$\boxed{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}}$$

$\Updownarrow$

$$\begin{aligned} x^2+y^2+z^2 &= (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 \\ &= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi \\ &= \rho^2\sin^2\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{1} + \rho^2\cos^2\phi \\ &= \rho^2(\sin^2\phi+\cos^2\phi) \end{aligned}$$

$\therefore\ x^2+y^2+z^2 = \rho^2$

และ $\tan\theta = \frac{y}{x}$

และ $z = \rho\cos\phi$

$\therefore\ \cos\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

ระบบพิกัดทรงกระบอก (r,θ,z)

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\longleftarrow$ $\longrightarrow$

ระบบพิกัดทรงกลม (ρ,θ,ϕ)

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$r\cancel{\cos\theta} = \rho\sin\phi\cancel{\cos\theta}$
$\theta = \theta$
$z = \rho\cos\phi$

$\Longleftrightarrow$ $\boxed{\begin{aligned} r &= \rho\sin\phi \\ \theta &= \theta \\ z &= \rho\cos\phi \end{aligned}}$

$\underbrace{x^2+y^2}_{r^2}+z^2 = \rho^2$ แต่ในระบบพิกัดทรงกระบอก $x^2+y^2=r^2$

$\therefore\ r^2+z^2=\rho^2 \Longleftrightarrow \rho = \sqrt{r^2+z^2}$

$\theta = \theta$

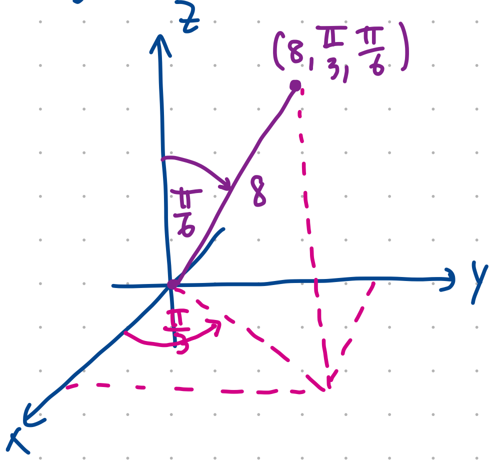
ในระบบพิกัดทรงกลม $\frac{y}{z} = \tan\phi\sin\theta$

จาก $y = r\sin\theta$ ได้ว่า $\frac{r\cancel{\sin\theta}}{z} = \tan\phi\cancel{\sin\theta}$

$\therefore\ \tan\phi = \frac{r}{z}$

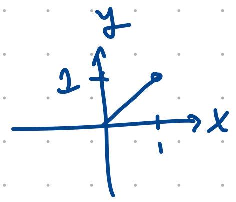
$$\boxed{\begin{aligned} \rho &= \sqrt{r^2+z^2} \\ \theta &= \theta \\ \tan\phi &= \frac{r}{z} \end{aligned}}$$

Ex $(8, \frac{\pi}{3}, \frac{\pi}{6})$ ที่เป็นพิกัดทรงกลมในพิกัดฉาก



$$x = \rho \sin\phi \cos\theta \iff x = 8 \sin\frac{\pi}{6} \cos\frac{\pi}{3} = 8 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2$$
$$y = \rho \sin\phi \sin\theta \iff y = 8 \sin\frac{\pi}{6} \sin\frac{\pi}{3} = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$$
$$z = \rho \cos\phi \iff z = 8 \cos\frac{\pi}{6} = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$$

$(8, \frac{\pi}{3}, \frac{\pi}{6})$ คือ $(2, 2\sqrt{3}, 4\sqrt{3})$



$(1, 1, \sqrt{2})$ เป็นพิกัดทรงกลม

$$\left.\begin{aligned} 1 &= \rho \sin\phi \cos\theta \\ 1 &= \rho \sin\phi \sin\theta \\ \sqrt{2} &= \rho \cos\phi \end{aligned}\right\}$$

$$x^2 + y^2 + z^2 = \rho^2$$
$$\rho = \sqrt{1+1+2}$$
$$\therefore \rho = 2$$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

เลือก $\theta = \frac{\pi}{4}$

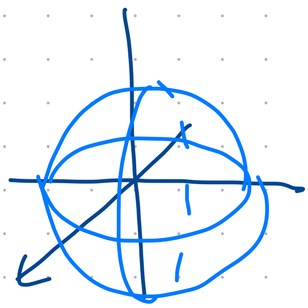
$$\therefore \sqrt{2} = 2\cos\phi$$
$$\cos\phi = \frac{\sqrt{2}}{2} \iff \phi = \frac{\pi}{4}$$

$\therefore (1, 1, \sqrt{2})$ คือ $(2, \frac{\pi}{4}, \frac{\pi}{4})$ ในระบบพิกัดทรงกลม.

Constant surface (ρ, θ, ϕ)

$\boxed{1}$ $\rho = 1$

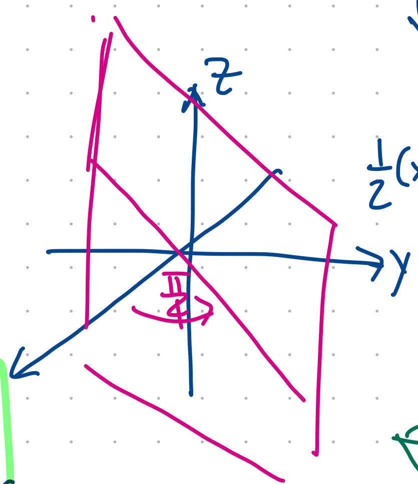
$$x^2 + y^2 + z^2 = 1$$



$\rho = \rho_0 \iff$ ผิวทรงกลม รัศมี ρ_0 หน่วย จุดศูนย์กลางอยู่ที่ (0,0,0)

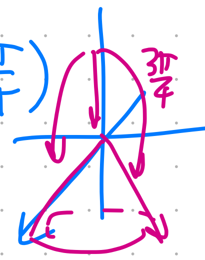
$\boxed{2}$ $\theta = \frac{\pi}{4}$

$$\tan\theta = \frac{y}{x}$$
$$\therefore \tan\frac{\pi}{4} = \frac{y}{x} \iff y = x$$

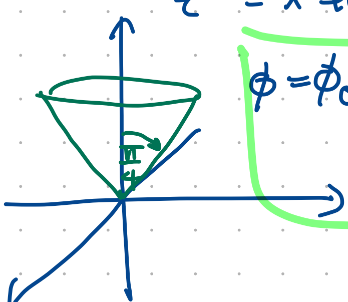


$\theta = \theta_0 \iff$ ระนาบตั้งฉากกับ ระนาบ xy ทำมุม θ_0 กับแกน x

$\boxed{3}$ $\phi = \frac{\pi}{4}$ $\quad (\phi = \frac{3\pi}{4})$



$$\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$\cos\frac{\pi}{4} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$\frac{\sqrt{2}}{2} = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad (z \geq 0)$$
$$\frac{1}{2}(x^2+y^2+z^2) = z^2$$
$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{z^2}{2}$$
$$z^2 = x^2 + y^2 \; ; \; z \geq 0$$



$\phi = \phi_0 \iff$ กรวยที่ทำมุม ϕ_0 จากแกน z

Constant surfaces

In **rectangular coordinates**, the surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \ Vz = z_0$$

where x_0, y_0, and z_0 are constants, are planes parallel to the yz-plane, xz-plane, and xy-plane, respectively.

In **cylindrical coordinates**, the surfaces represented by equations of the form

$$r = r_0, \quad \theta = \theta_0, \quad z = z_0$$

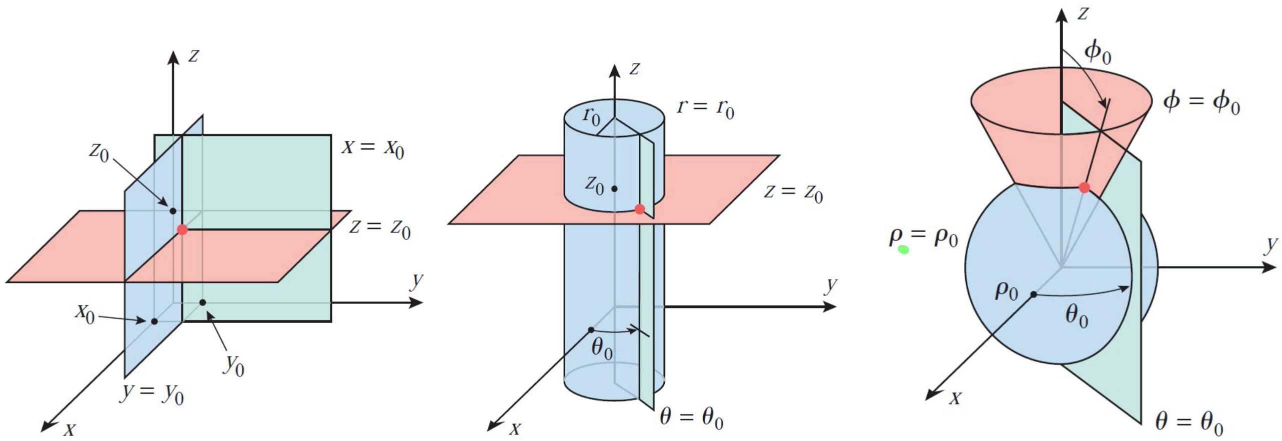
where r_0, θ_0, and z_0 are constants,

- The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis.
- The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.
- The surface $z = z_0$ is a horizontal plane.

In **spherical coordinates**, the surfaces represented by equations of the form

$$\rho = \rho_0, \quad \theta = \theta_0, \quad \phi = \phi_0$$

- The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ to be nonnegative, this is a sphere of radius ρ centered at the origin.
- As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.
- The surface $\phi = \phi_0$ consists of all points from which a line segment to the origin makes an angle of ϕ_0 with the positive z-axis. If $0 < \phi_0 < \pi/2$, this will be the nappe of a cone opening up, while if $\pi/2 < \phi_0 < \phi$, this will be the nappe of a cone opening down. (If $\phi_0 = \pi/2$, then the cone is flat, and the surface is the xy-plane.)



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 832-833

พิกัดทรงกระบอก $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ | พิกัดทรงกลม $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

Equations of Surfaces in Cylindrical and Spherical Coordinates

Surfaces of revolution about the z-axis of a rectangular coordinate system usually have simpler equations in cylindrical coordinates than in rectangular coordinates.

Example 6. Find equations of the ~~paraboloid~~ cone $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.

พิกัดทรงกระบอก

$$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$z = r$$

พิกัดทรงกลม

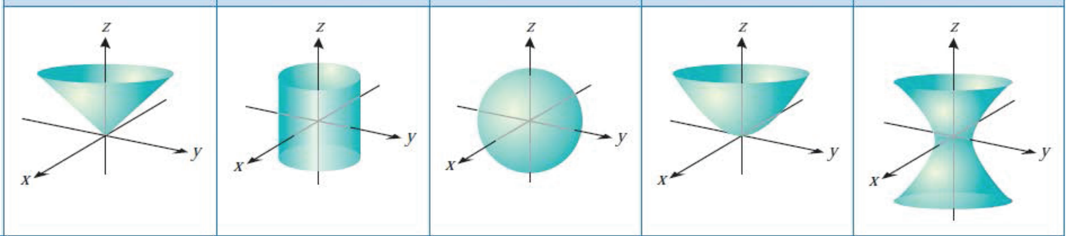
$$z = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$= \sqrt{\rho^2\sin^2\phi\,(\cos^2\theta + \sin^2\theta)}$$

$$\therefore z = \rho\sin\phi$$

แต่ $z = \rho\cos\phi = \rho\sin\phi \iff \cos\phi = \sin\phi$

$$\boxed{\phi = \frac{\pi}{4}}$$



	CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
RECTANGULAR	$z = \sqrt{x^2+y^2}$	$x^2+y^2=1$	$x^2+y^2+z^2=1$	$z = x^2+y^2$	$x^2+y^2-z^2=1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi\csc^2\phi$	$\rho^2 = -\sec 2\phi$

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 835

Triple Integrals in Cylindrical Coordinates

Recall that in rectangular coordinates the triple integral of a continuous function f over a solid region G is defined as

$$\iiint_G f(x,y,z)\,dV = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

where ΔV_k denotes the volume of a rectangular parallelepiped interior to G and (x_k^*, y_k^*, z_k^*) is a point in this parallelepiped. Triple integrals in cylindrical and spherical coordinates are defined similarly, except that the region G is divided not into rectangular parallelepipeds but into regions more appropriate to these coordinate systems.

In cylindrical coordinates, the simplest equations are of the form

$$r = \text{constant}, \quad \theta = \text{constant}, \quad z = \text{constant}$$

These surfaces can be paired up to determine solids called *cylindrical wedges* or *cylindrical elements of volume*. To be precise, a cylindrical wedge is a solid enclosed between six surfaces of the following form: