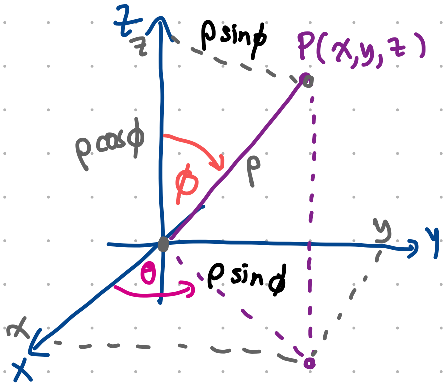
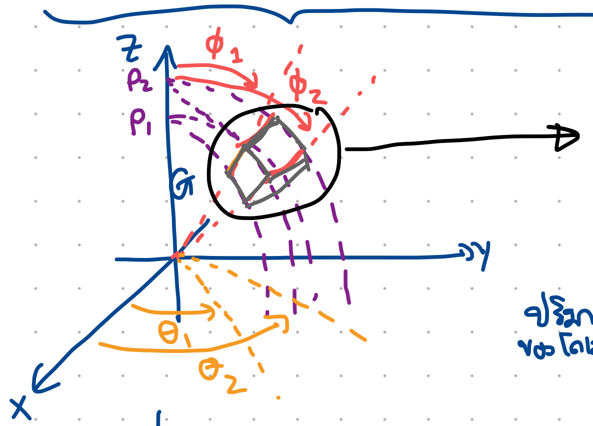
Triple integral in spherical coordinate

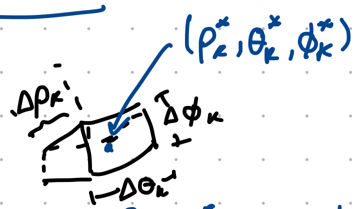


$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}}$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \tan\theta &= \frac{y}{x} \\ \cos\phi &= \frac{z}{\rho} \end{aligned}}$$





$$\Delta V_k^* = \rho_k^{*2}\sin\phi_k^*\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$$

ปริมาตร n ชิ้น ของโดเมน G $\displaystyle = \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\,\Delta V_k$

$$= \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\,\rho_k^{*2}\sin\phi_k^*\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$$

Take limit : $\displaystyle = \lim_{n\to\infty}\sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\,\rho_k^{*2}\sin\phi_k^*\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$

$$= \iiint_G f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$\therefore$ Triple integral

$$\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

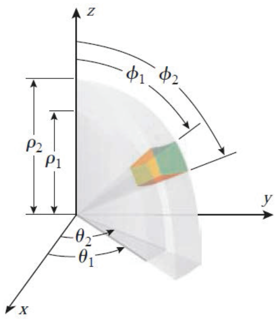
Triple Integrals in Spherical Coordinates

In spherical coordinates, the simplest equations are of the form

$$\rho = \text{constant}, \quad \theta = \text{constant}, \quad \phi = \text{constant}$$

These surfaces can be paired up to determine solids called *spherical wedges* or *spherical elements of volume*. To be precise, a spherical wedge is a solid enclosed between six surfaces of the following form:

two spheres	$\rho = \rho_1,\ \rho = \rho_2 \quad (\rho_1 < \rho_2)$
two vertical half-planes	$\theta = \theta_1,\ \theta = \theta_2 \quad (\theta_1 < \theta_2)$
nappes of two circular cones	$\phi = \phi_1,\ \phi = \phi_2 \quad (\phi_1 < \phi_2)$



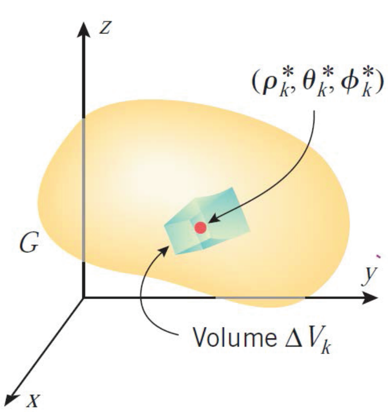
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1051

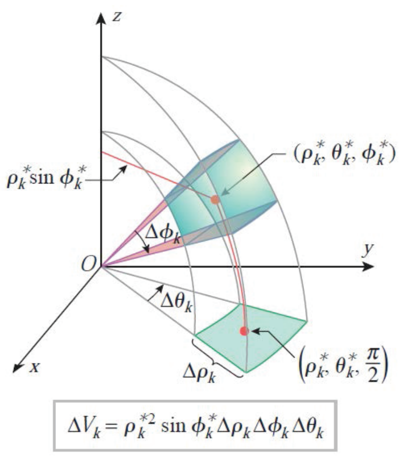
If G is a solid region in three-dimensional space, then the triple integral over G of a continuous function $f(\rho, \theta, \phi)$ in spherical coordinates is similar in definition to the triple integral in cylindrical coordinates, except that the solid G is partitioned into spherical wedges by a three-dimensional grid consisting of spheres centered at the origin, half-planes hinged on the z-axis, and nappes of right circular cones with vertices at the origin and lines of symmetry along the z-axis.

The defining equation of a triple integral in spherical coordinates is

$$\iiint_G f(\rho, \theta, \phi)\, dV \;=\; \lim_{n \to +\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*) \Delta V_k$$

where ΔV_k is the volume of the kth spherical wedge that is interior to G, $(\rho_k^*, \theta_k^*, \phi_k^*)$ is an arbitrary point in this wedge, and n increases in such a way that the dimensions of each interior spherical wedge tend to zero.





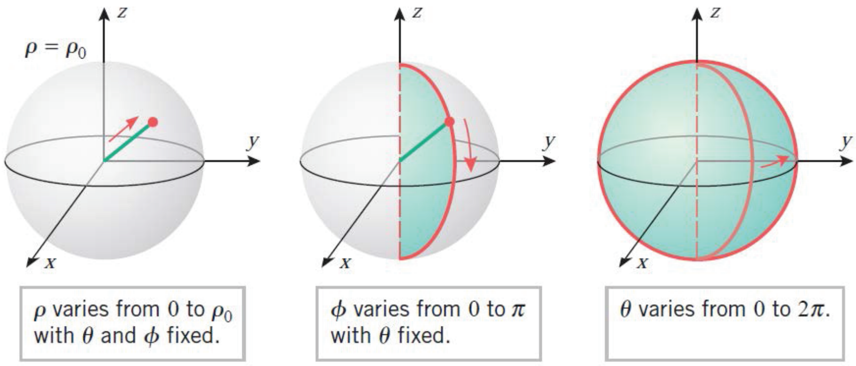
$$\iiint_G f(\rho,\theta,\phi)\,dV \;=\; \lim_{n\to+\infty}\sum_{k=1}^{n} f(\rho_k^*,\theta_k^*,\phi_k^*)(\rho_k^*)^2\sin\phi_k^*\Delta\rho_k\Delta\phi_k\Delta\theta_k.$$

which suggests that a triple integral in cylindrical coordinates can be evaluated as an iterated integral of the form

$$\iiint_G f(\rho,\theta,\phi)\,dV \;=\; \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Suppose that we want to integrate $f(\rho,\theta,\phi)$ over the spherical solid G enclosed by the sphere $\rho=\rho_0$. The basic idea is to choose the limits of integration so that every point of the solid is accounted for in the integration process. The below figure illustrates one way of doing this. Holding θ and ϕ fixed for the first integration, we let ρ vary from 0 to ρ_0. This covers a radial line from the origin to the surface of the sphere. Next, keeping θ fixed, we let ϕ vary from 0 to π so that the radial line sweeps out a fan-shaped region. Finally, we let θ vary from 0 to 2π so that the fan-shaped region makes a complete revolution, thereby sweeping out the entire sphere. Thus, the triple integral of$f(\rho,\theta,\phi)$ over the spherical solid G can be evaluated by writing

$$\iiint_G f(\rho,\theta,\phi)\,dV \;=\; \int_0^{2\pi}\int_0^{\pi}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$



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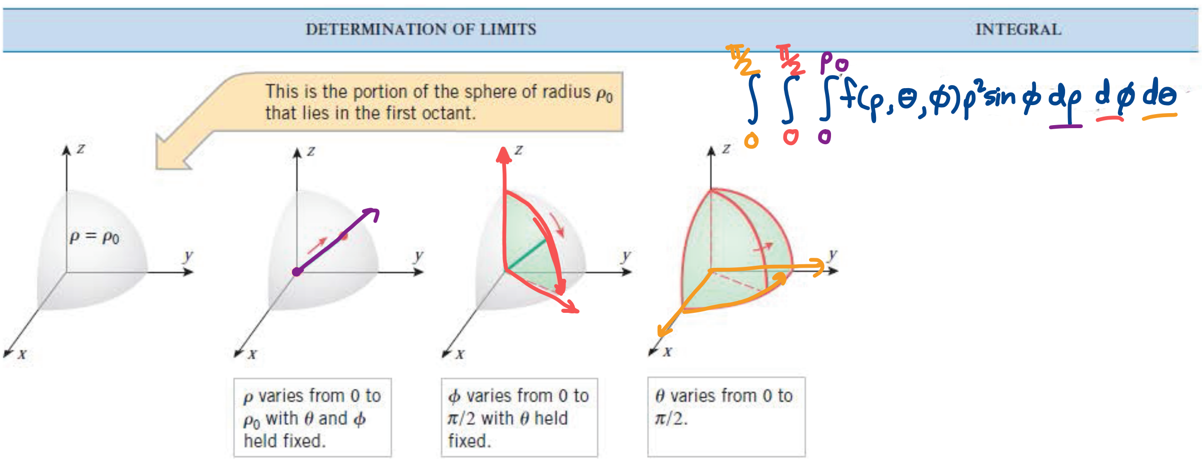
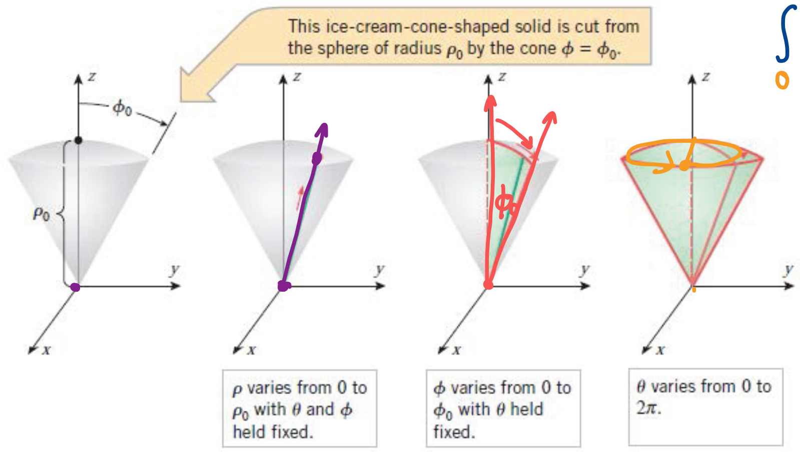
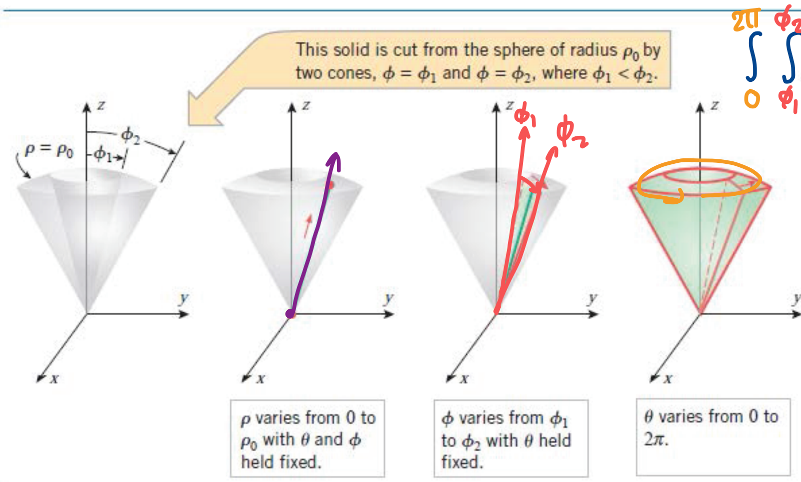


$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



ระนาบ $z = a$

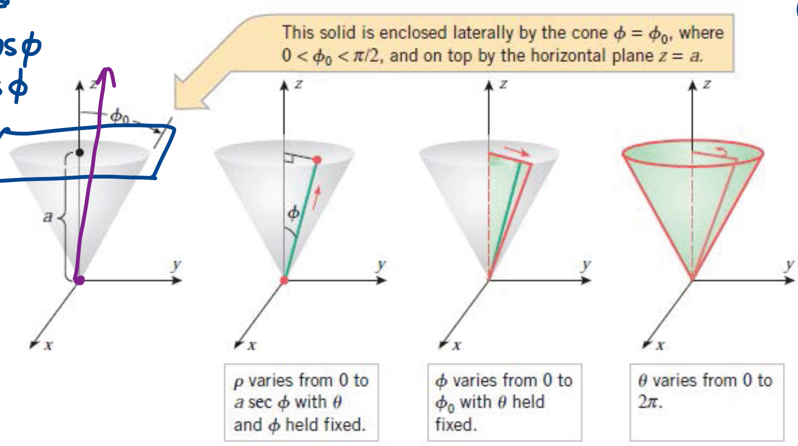
จาก $z = \rho\cos\phi$

$a = \rho\cos\phi$

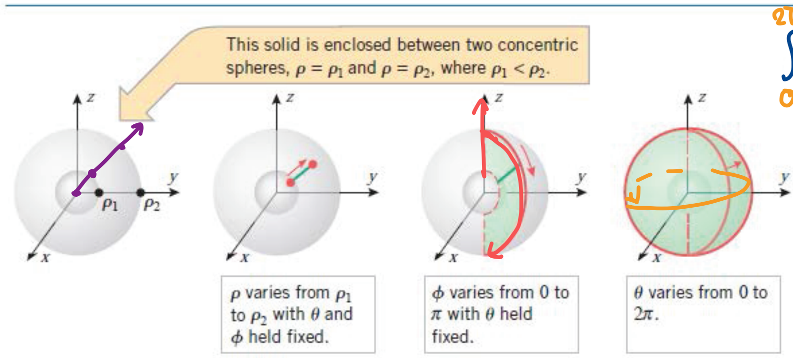
$\therefore \rho = \dfrac{a}{\cos\phi}$

$\therefore \rho = a\sec\phi$

$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{a\sec\phi} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



$$\int_0^{2\pi}\int_0^{\pi}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



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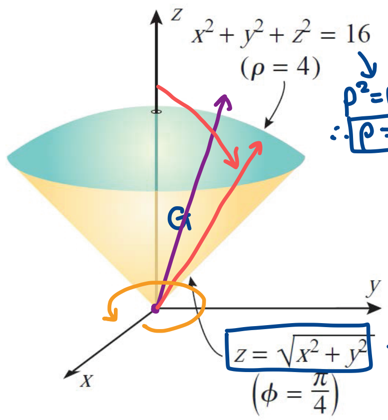
$\sqrt{x^2+y^2+z^2} = \rho$

$\tan\theta = \frac{y}{x}$

$\cos\phi = \frac{z}{\rho}$

(set $f(\rho,\theta,\phi) = 1$)

Example 9. Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.



$\rho^2 = 16$

$\therefore \boxed{\rho = 4}$

$\rightarrow z = \sqrt{x^2+y^2}$

$z = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)}$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$\therefore \phi = \frac{\pi}{4}$

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=4} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{64}{3}\sin\phi\, d\phi\, d\theta$$

$$= \frac{64}{3}\int_0^{2\pi}[-\cos\phi]\Big|_{\phi=0}^{\pi/4} d\theta$$

$$= \frac{64}{3}\int_0^{2\pi}\left(1-\frac{\sqrt{2}}{2}\right) d\theta$$

$$= \frac{64}{3}(2\pi)\left(1-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{64}{3}(2-\sqrt{2})\pi \quad \#$$

From:Calculus Early Transcendentals, 10th edition,Howard Anton,Irl C. Beven, Stephen Deavis, page 1054

Converting Triple Integrals from Rectangular to Spherical Coordinates

Triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. The two integrals are related by the equation

$$\iiint_G f(x,y,z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

Example 10. Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

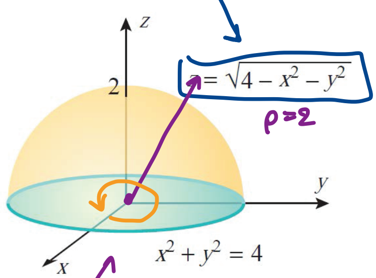
$z^2 = 4 - x^2 - y^2$

$x^2 + y^2 + z^2 = 4$

$\rho^2 = 4 \Leftrightarrow \rho = 2$

① แปลง $z^2\sqrt{x^2+y^2+z^2}$: $\rho = \sqrt{x^2+y^2+z^2}$, $z = \rho\cos\phi$

$\therefore z^2\sqrt{x^2+y^2+z^2} = (\rho\cos\phi)^2\cdot\rho = \rho^3\cos^2\phi$



② เขียนขอบเขต

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} \boxed{z^2\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho^3\cos^2\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^5\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^6}{6}\cos^2\phi\sin\phi\Big|_{\rho=0}^{2} d\phi\, d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\int_0^{\pi/2} \cos^2\phi\sin\phi\, d\phi\, d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\left[-\frac{\cos^3\phi}{3}\right]_{\phi=0}^{\pi/2} d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\left[0+\frac{1}{3}\right] d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\frac{1}{3}\, d\theta$$

$$= \frac{64\pi}{9} \quad \#$$

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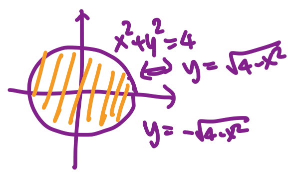


(หา) $\int \cos^2\phi\sin\phi\, d\phi$

$u = \cos\phi$; $du = -\sin\phi\, d\phi$

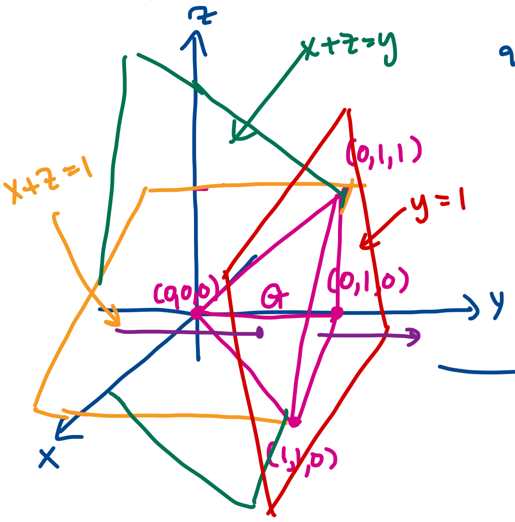
$\therefore \int \cos^2\phi\sin\phi\, d\phi = -\int u^2\, du$

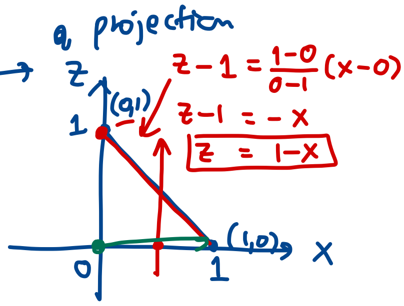
$= -\frac{u^3}{3} + C = -\frac{\cos^3\phi}{3} + C$

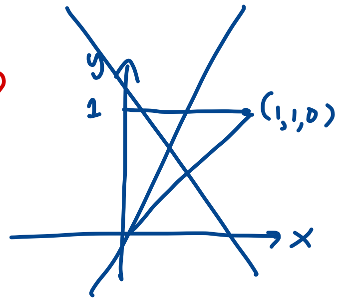
แบบฝึกหัดเพิ่มเติม Multiple integrals

1. กำหนดให้ G เป็นทรงตันที่มีจุดยอดเป็น $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ และ $(0,1,1)$ ปิดล้อมด้วยระนาบ $x+z=y, x+z=1, y=1$ จงหาลิมิตของการอินทิเกรตของอินทิกรัลสามชั้นของฟังก์ชัน $F(x,y,z)$ บน G โดยให้ลำดับของการอินทิเกรตเป็น $dy\,dz\,dx$.

ตอบ $\iiint_G F(x,y,z)\,dV = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x,y,z)\,dy\,dz\,dx$

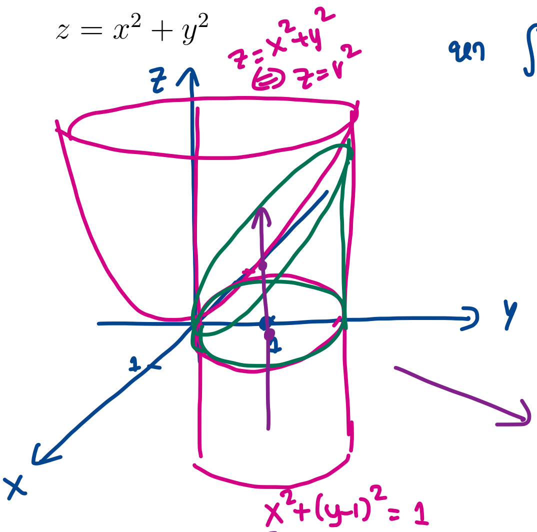


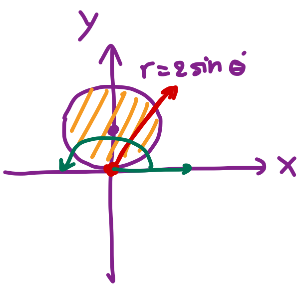




2. จงหาลิมิตของการอินทิเกรตในระบบพิกัดทรงกระบอกของฟังก์ชัน $f(r,\theta,z)$ บนบริเวณ G ซึ่งปิดล้อมด้านล่างด้วยระนาบ $z=0$ ด้านข้างด้วยทรงกระบอก $x^2+(y-1)^2=1$ และด้านบนด้วยพาราโบลอยด์ $z=x^2+y^2$

ตอบ $\iiint_G f(r,\theta,z)\,dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r,\theta,z)\,r\,dz\,dr\,d\theta$





$x^2+y^2=r^2$

$y=r\sin\theta$

$x^2+(y-1)^2=1$

$x^2+y^2-2y+\cancel{1}=\cancel{1}$

$r^2-2y=0$

$r^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$

3. จงหาปริมาตรของทรงตันที่ปิดล้อมด้านล่างด้วยทรงกลม $\rho = 2\cos\phi$ และปิดล้อมด้านบนด้วยกรวย $z = \sqrt{x^2+y^2}$

4. จงหาปริมาตรของทรงตันที่อยู่ภายในทรงกลม $\rho \leq a$ อยู่ระหว่างกรวย $\phi = \dfrac{\pi}{3}$ และ $\phi = \dfrac{2\pi}{3}$ ในอัฐภาคที่ 1 (เมื่อ $x, y, z \geq 0$)