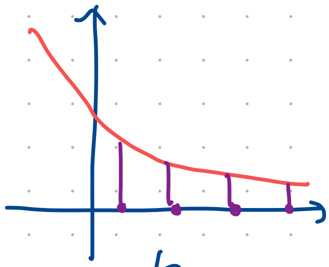
· Integral test $\sum_{n=a}^{\infty} a_n$ ถ้า $f(n) = a_n \quad \forall n \geq a$

$\Downarrow$

พิจารณาว่า $\int_a^{+\infty} f(x)\,dx$ Converges / diverges.



**Converges**

- อนุกรมเรขาคณิต $\sum_{n=1}^{\infty} ar^n \;;\; |r| < 1$
- อนุกรม telescopic $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
- อนุกรม p $\sum_{n=1}^{\infty} \frac{1}{n^p} \;;\; p > 1$

**Diverges.**

- อนุกรมเรขาคณิต $\sum_{n=1}^{\infty} ar^n \;;\; |r| \geq 1$
- อนุกรมอื่นๆ $\sum_{n=1}^{\infty} a_n$ ซึ่ง $\lim_{n\to\infty} a_n \neq 0$
- อนุกรม p $\sum_{n=1}^{\infty} \frac{1}{n^p} \;;\; p \leq 1$
  - $\downarrow$ harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

3. $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

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4. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

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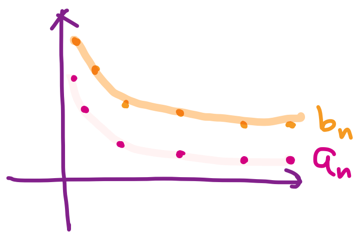
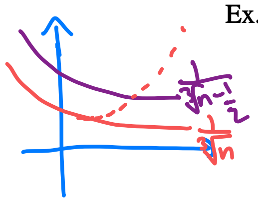
### 4.4.2 The Comparison Test

**Theorem 4.8.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms and suppose that and* $a_1 \le b_1, a_2 \le b_2, a_3 \le b_3, ..., a_k \le b_k, ...$

1. *If $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} a_n$ converge.*
2. *If $\sum_{n=1}^{\infty} a_n$ diverge, then $\sum_{n=1}^{\infty} b_n$ diverge.*




**Ex. 4.8.** Determine whether the following series converges or diverges.




1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} - \frac{1}{2}}$

จาก
$\sqrt[3]{n} \geq \sqrt[3]{n} - \frac{1}{2}$
$\frac{1}{\sqrt[3]{n}-\frac{1}{2}} \geq \frac{1}{\sqrt[3]{n}}$ → $\frac{1}{n^{\frac{1}{3}}}$
CHECK
ตัวลูก อนุกรม p
$(p = \frac{1}{3} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ div)

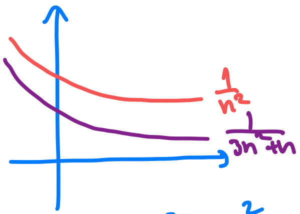
เนื่องจาก $a_n = \frac{1}{\sqrt[3]{n}} \le \frac{1}{\sqrt[3]{n} - \frac{1}{2}} = b_n \quad \forall n \geq 1$

และเราทราบว่า $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges เนื่องจาก เป็นอนุกรม p ซึ่ง $p = \frac{1}{3} < 1$

$\therefore$ โดย comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} - \frac{1}{2}}$ diverges. #

อนุกรม con/di ___






$3n^2+n \geqslant 3n^2 \geqslant n^2$

$\Leftrightarrow \frac{1}{n^2} \geqslant \frac{1}{3n^2} \geqslant \frac{1}{3n^2+n} \quad \forall n \geqslant 1$

เดา CHECK

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. $p = 2 > 1$

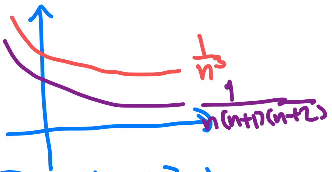
2. $\sum_{n=1}^{\infty} \frac{1}{3n^2+n}$

พิจารณา $a_n = \frac{1}{3n^2+n} \leqslant \frac{1}{n^2} = b_n \quad \forall n \geqslant 1$

เนื่องจาก $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges โดยทดสอบอนุกรม p · ซึ่ง $p = 2 > 1$

∴ โดย comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{3n^2+n}$ converges #

เลือกอนุกรม ___ ซึ่ง con/di ทราบ ___




ทด $(n+1 \geqslant n,\ n+2 \geqslant n)$

$n(n+1)(n+2) \geqslant n \cdot n \cdot n = n^3 \quad \forall n \geqslant 1$

$\frac{1}{n^3} \geqslant \frac{1}{n(n+1)(n+2)}$ CHECK

เดา

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. (p-series ; $p = 3 > 1$)

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

พิจารณา $a_n = \frac{1}{n(n+1)(n+2)} \leqslant \frac{1}{n^3} = b_n \quad \forall n \geqslant 1$

เนื่องจาก $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges จาก p-series s.t. $p = 3 > 1$

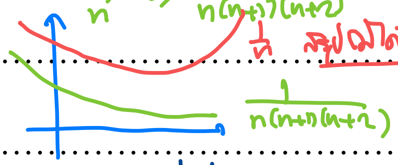
∴ โดย comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converges #

ทด telescopic series

$\frac{1}{n(n+1)} \geqslant \frac{1}{n(n+1)(n+2)}$

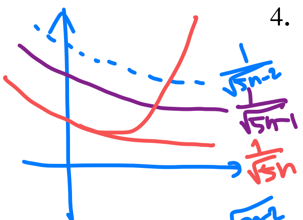

$n(n+1)(n+2) \geqslant n$

$\frac{1}{n} \geqslant \frac{1}{n(n+1)(n+2)}$




เลือก $b_n = \frac{1}{n(n+1)}$ จาก $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ conv. จาก telescopic.

∴ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ conv. #




$\sqrt{5n} \geqslant \sqrt{5n-1} \geqslant \sqrt{5n-2} \quad \forall n \geqslant 1$

$\frac{1}{\sqrt{5n-1}} \geqslant \frac{1}{\sqrt{5n}}$

CHECK เดา

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n}}$ div

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n-1}}$

พิจารณา $a_n = \frac{1}{\sqrt{5n}} \leqslant \frac{1}{\sqrt{5n-1}} = b_n \quad \forall n \geqslant 1$

จาก $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{5n}}$ diverges จากทดสอบ p-series $p = \frac{1}{2} < 1$

∴ โดย comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n-1}}$ diverges #

### 4.4.3 The Limit Comparison Test

ทดสอบ $a_n$ โดยหารด้วย $b_n$ แล้วดูลิมิต

> Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms and define $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$.
>
> 1. If $\rho$ is finite and $\rho > 0$ , then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
> 2. If $\rho = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
> 3. If $\rho = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

**Ex. 4.9.** Determine whether the following series converges or diverges by using the limit comparison test .

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ $a_n$

เลือก $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $b_n$ (diverges เนื่องจากเป็นอนุกรม $p = \frac{1}{2} < 1$ )

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}}+2}$$

$$= \lim_{n\to\infty} \frac{\frac{n^{1/2}}{n^{1/2}}}{\frac{n^{1/2}}{n^{1/2}} + \frac{2}{n^{1/2}}} = 1$$

ดังนั้น $\rho = \lim_{n\to\infty} \frac{a_n}{b_n} = 1$ และ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges ( p-series s.t. $p<1$ )

โดย limit comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ diverges #

2. $\sum_{n=1}^{\infty} \frac{n+2}{n^2}$ $a_n$

เลือก $\sum_{n=1}^{\infty} \frac{n+2}{n}$

( $\sum_{n=1}^{\infty} \frac{n+2}{n}$ div ) ↳ $\lim_{n\to\infty} \frac{n+2}{n} = 1$

$$\rho = \lim_{n\to\infty} \frac{\frac{n+2}{n^2}}{\frac{n+2}{n}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot n = \lim_{n\to\infty} \frac{1}{n} = 0$$ (สรุปไม่ได้)

เลือก $\sum \frac{1}{n^2}$ conv. (p-series $p>1$)

$\rho = \lim_{n\to\infty} \frac{\frac{n+2}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} n+2 = +\infty$ (สรุปไม่ได้ ?)

เลือกใหม่: เลือก $\sum_{n=1}^{\infty} \frac{1}{n}$ (div harmonic series)

$$\rho = \lim_{n\to\infty} \frac{\frac{n+2}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+2n}{n^2} = 1$$ ; ดังนั้น $\rho = 1$ และ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

โดย limit comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{n+2}{n^2}$ diverges . #

3. $\sum_{n=1}^{\infty} \frac{3n^3-4n^2+5}{n^5-n^3+2}$ $a_n$

เลือก $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (เป็นอนุกรม $p = 2 > 1$)

$$\rho = \lim_{n\to\infty} \frac{\frac{3n^3-4n^2+5}{n^5-n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3-4n^2+5}{n^5-n^3+2} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{3n^5-4n^4+5n^2}{n^5-n^3+2} = 3 > 0$$

เนื่องจาก $\rho = 3$ และ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ( p-series ซึ่ง $p=2>1$ )

∴ โดย limit comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{3n^3-4n^2+5}{n^5-n^3+2}$ converges #

สมมติเลือก $\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv.

$$\rho = \lim_{n\to\infty} \frac{3n^3-4n^2+5}{n^5-n^3+2} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{3n^6-4n^5+5n^3}{n^5-n^3+2} = +\infty$$ ← สรุปไม่ได้ ?

เลือก $\sum_{n=1}^{\infty} \frac{2}{n^2}$ ได้

4. $\displaystyle\sum_{n=1}^{\infty} \overbrace{\frac{2n^2+n}{n^4+\sqrt{n}}}^{a_n}$ เลือก $\displaystyle\sum_{n=1}^{\infty} \overbrace{\frac{1}{n^2}}^{b_n}$ (conv ตาม p-series $p=2>1$)

$$\rho = \lim_{n\to\infty} \frac{\frac{2n^2+n}{n^4+\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2+n}{n^4+\sqrt{n}} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{2n^4+n^3}{n^4+\sqrt{n}} = 2$$

จาก $\rho = 2$ และ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series ซึ่ง $p=2>1$)

โดย limit comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{2n^2+n}{n^4+\sqrt{n}}$ converges.

### 4.4.4 The Ratio Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series with positive terms and $\rho = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $\rho < 1$, then the series converges.
2. If $\rho > 1$ or $\rho = \infty$ , then the series diverges.
3. If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

**Ex. 4.10.** Determine whether the following series converges or diverges by using the ratio test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

Ex พิจารณา $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $a_n$

เลือก $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $b_n$ (conv. เพราะอนุกรมเรขาคณิต $|r| = \frac{1}{2} < 1$)

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^n-1}\cdot\frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{\cancel{\frac{2^n}{2^n}}}{\cancel{\frac{2^n}{2^n}} - \cancel{\frac{1}{2^n}}^{\to 0}} = 1$$

จาก $\rho = 1$ และ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (เพราะอนุกรมเรขาคณิต ซึ่ง $|r| = \frac{1}{2} < 1$)

โดย limit comparison test ได้ว่า $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges #

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{5^n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n-1}$

### 4.4.5 The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and $\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n}$.

1. If $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\rho > 1$ or $\rho = \infty$ , $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $\rho = 1$ , the series may converge or diverge, so that another test must be tried.

**Ex. 4.11.** Determine whether the following series converges or diverges by using the root test.

1. $\sum_{n=2}^{\infty} (\frac{9n-1}{3n+2})^n$

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2. $\sum_{n=1}^{\infty} \frac{1}{(\ln(2n+3))^n}$

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3. $\sum_{n=1}^{\infty}(n^{1/n}-1)^n$

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4. $\sum_{n=1}^{\infty}(\frac{n+1}{n})^{n^2}$

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