# Infinite Series $\sum_{n=1}^{\infty} a_n$

- Integral test
  : $f(x)$ s.t. $f(n)=a_n \quad \forall n \geq a$
  - $f(x)$ decreasing on $[a,\infty)$ + cont

  พิจารณา $\int_a^{+\infty} f(x)\,dx$ conv/div $\Leftrightarrow \sum_{n=1}^{\infty} a_n$ conv/div ตาม $\int_a^{\infty} f(x)\,dx$

- Comparison test
  $\Updownarrow$
- Limit comparison test

C
- Geom. Series $\sum_{n=1}^{\infty} ar^n$; $|r|<1$
- Telescopic $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
- p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$; $p>1$
- $\sum_{n=1}^{\infty} \frac{1}{n!}$

D

Convergent Series

D
- Geom. Series $\sum_{n=1}^{\infty} ar^n$; $|r|\geq 1$
- $\sum_{n=1}^{\infty} a_n$ s.t. $\lim_{n\to\infty} a_n \neq 0$
- p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$; $p \leq 1$
- Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$

D

Divergent series

- ratio test : $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$
  - $\rho<1$ conv.
  - $\rho>1$, $\rho=\infty$ ; div
  - $\rho=1$ $\times$

- root test : $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$
  - $\rho<1$ conv.
  - $\rho>1$, $\rho=\infty$ div
  - $\rho=1$ $\times$

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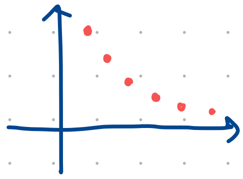
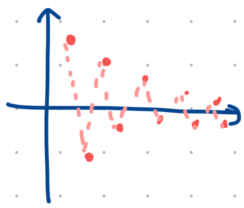
## Alternating Series



Ex $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$



Q: $\sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \frac{1}{2^n}$ ทดสอบแบบนี้ได้ไหม

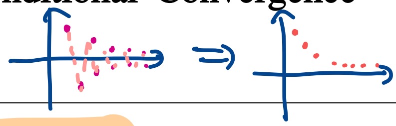
$\rightarrow \sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \frac{1}{2^n} = -\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \dots$



ใช้วิธีนี้ทดสอบไม่ได้.

## 4.5 Alternating Series, Absolute and Conditional Convergence

### 4.5.1 Convergence Test for Alternating Series



Leibniz's Theorem

An alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ ( $a_n > 0$ for all $n$ ) converges if ~~and only if~~

1. $a_n \geq a_{n+1}$ for all $n$ $\Leftrightarrow$ แสดงว่าเป็น $\{a_n\}$ เป็นลำดับลด $\Leftrightarrow a_n - a_{n+1} \geq 0$ หรือ $\frac{a_n}{a_{n+1}} \geq 1$ หรือ ให้ $f(x)$ ซึ่ง $f(n) = a_n$ $\forall n$ check $f'(x) \leq 0$
2. $\lim_{n\to\infty} a_n = 0$

**Ex. 4.12.** Determine whether the following alternating series converges or diverges.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ $a_n$ $\left[\sum_{n=1}^{\infty}\frac{1}{n}\text{ div. จาก harmonic series}\right]$

ข้อ 2

(1) Check $a_n \geq a_{n+1}$ : เรารู้ว่า $n+1 \geq n \Leftrightarrow \frac{1}{n} \geq \frac{1}{n+1}$ $\forall n \geq 1$

$\therefore a_n \geq a_{n+1}$

$\left( a_n - a_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0 \right)$

(2) Check $\lim_{n\to\infty}\frac{1}{n} = 0$

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges #

$1, 2, 3, 4, 5, 6, 7, \dots \rightarrow \{1, 3, 5, 7, \dots\}$, $\{2, 4, 6, \dots\}$

2. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n+1}$ $\left[\sum_{n=1}^{\infty}\frac{n}{2n+1}\text{ check: } \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} \neq 0\right]$

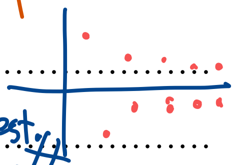
พิจารณา $\lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} \neq 0$

Check $\left\{(-1)^n\frac{n}{2n+1}\right\}$ conv / div ?

ถ้า $n$ เป็นเลขคู่ : $\lim_{n\to\infty}(-1)^n\frac{n}{2n+1} = \frac{1}{2}$

ถ้า $n$ เป็นเลขคี่ : $\lim_{n\to\infty}(-1)^n\frac{n}{2n+1} = -\frac{1}{2}$

subsequence oscillate ระหว่าง $\frac{1}{2}, -\frac{1}{2}$



$\therefore \left\{(-1)^n\frac{n}{2n+1}\right\}$ diverges

$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{n}{2n+1}$ diverges โดย divergent test #

3. $\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+n}$ $\left[\sum_{n=1}^{\infty}\frac{n+3}{n^2+n}\text{ ~~conv~~ div : เลือก }\sum_{n=1}^{\infty}\frac{1}{n}\text{ div และ }\lim_{n\to\infty}\frac{n+3}{n^2+n}\Big/\frac{1}{n} = 1\right]$

พิจารณา ① $\{a_n\}$ เป็นลำดับลดไหม? $\Rightarrow$ เป็น แน่นอน

พิจารณา $f(x) = \frac{x+3}{x^2+x}$ ซึ่ง $f(n) = a_n$ $\forall n \geq 1$

$f'(x) = \frac{(x^2+x) - (x+3)(2x+1)}{(x^2+x)^2} = \frac{x^2+x-(2x^2+7x+3)}{(x^2+x)^2}$

$= \frac{-x^2-6x-3}{(x^2+x)^2} = -\frac{(x^2+6x+3)}{(x^2+x)^2} < 0$ $\forall x \geq 1$ ($x^2+6x+3 > 0$, $(x^2+x)^2 > 0$)

② $\lim_{n\to\infty}\frac{n+3}{n^2+n} = 0$. $\therefore \sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+n}$ converges #

การลู่เข้าแบบสัมบูรณ์ การลู่เข้าแบบมีเงื่อนไข

### 4.5.2 Absolute Convergence and Conditionally Convergence

- A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

- A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if the series $\sum_{n=1}^{\infty} |a_n|$ diverges but the series $\sum_{n=1}^{\infty} a_n$ converges.

$\sum_{n=1}^{\infty} a_n$ — div. / cond. conv. / abs. conv.

Check $\sum_{n=1}^{\infty} a_n$

$\boxed{1}$ $\sum_{n=1}^{\infty} |a_n|$ conv? $\xrightarrow{Y}$ conv. abs.

$\downarrow N$

$\boxed{2}$ $\sum_{n=1}^{\infty} a_n$ conv? $\xrightarrow{Y}$ conv. cond.

$\downarrow N$

diverges

**Remarks:**

1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Converges absolutely $\Rightarrow$ Converges

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

**Ex. 4.13.** Determine that the following series converge absolutely or converge conditionally.

1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n}$

   พิจารณา $\boxed{1}$ $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{2^n}\right| = \sum_{n=1}^{\infty} \frac{1}{2^n}$ เป็นอนุกรมเรขาคณิต $|r| = \frac{1}{2} < 1$

   $\therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

   $\therefore \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n}$ converges absolutely #

2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

   พิจารณา $\boxed{1}$ $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges โดย harmonic series.

   (ไม่ conv. absolutely!)

   $\boxed{2}$ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ Converges (Ex 4.12 ข้อ 1)

   $\therefore \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges conditionally #

3. $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{n}{n^2+1})^n$

$\boxed{1}$ $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{n}{n^2+1}\right)^n\right| = \sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)^n$

check root-test : $\rho = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n^2+1}\right)^n} = \lim_{n\to\infty}\frac{n}{n^2+1} = 0 < 1$

$\therefore \sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)^n$ conv. โดยใช้ root test

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n^2+1}\right)^n$ converges absolutely. #

4. $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \ldots$

$\boxed{1}$ พิจารณา $\sum_{n=1}^{\infty}|a_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots$

เป็นอนุกรมเรขาคณิตที่มี $|r| = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty}|a_n|$ converges

$\therefore$ อนุกรม $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \ldots$ converges absolutely #

$\Updownarrow$

$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \ldots$ converges #

5. $\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n!}$

$\boxed{1}$ พิจารณา $\sum_{n=1}^{\infty}\left|(-1)^n\frac{3^n}{n!}\right| = \sum_{n=1}^{\infty}\frac{3^n}{n!}$

พิจารณา $\rho = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!} \Big/ \frac{3^n}{n!} = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}$

$= \lim_{n\to\infty}\frac{3\cdot\cancel{3^n}\cancel{n!}}{(n+1)\cancel{n!}\cdot\cancel{3^n}} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$

โดย ratio test ได้ว่า $\sum_{n=1}^{\infty}\frac{3^n}{n!}$ converges

$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n!}$ converges absolutely. #

6. $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$

$-1 \leq \cos n \leq 1$

$a_n = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ conv. $= b_n$

$\boxed{1}$ พิจารณา $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$

โดย comparison test เลือก $a_n = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2} = b_n$

และ $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges จากการทดสอบ p-series เมื่อ $p = 2 > 1$

$\therefore$ โดย comparison test ได้ว่า $\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ converges.

$\therefore \sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges absolutely #

## Ratio Test for Absolute Convergence

**Theorem:** Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms and $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

- If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and therefore converges.
- If $\rho > 1$ or $\rho = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.

**Ex. 4.14.** Use the ratio test for absolute convergence to determine whether the series converges.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ $a_n$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot 2^{n+1}}{(n+1)!} \Big/ \frac{(-1)^n \cdot 2^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{2 \cdot \cancel{2^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} \right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

โดย Ratio test ของ absolute conv. check, ได้ว่า $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ converges absolutely

นั่นคือ $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ converges. #

2. $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{3^n}$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (2(n+1)-1)!}{3^{n+1}} \Big/ \frac{(-1)^n (2n-1)!}{3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{(2n+1)!}{3^{n+1}} \cdot \frac{3^n}{(2n-1)!} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n)\cancel{(2n-1)!} \cdot \cancel{3^n}}{3 \cdot \cancel{3^n} \cdot \cancel{(2n-1)!}} = +\infty$$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{3^n}$ diverges #

โดย ratio test ของ absolute conv.

ประโยชน์ของ absolutely convergence

rearrangement

**Theorem** ถ้า $\sum_{n=1}^{\infty} a_n$ conv. absolutely และ $b_1, b_2, \ldots, b_n, \ldots$ เป็นการสลับลำดับของ $a_1, a_2, \ldots, a_n, \ldots$ แล้ว อนุกรม $\sum_{n=1}^{\infty} b_n$ conv. absolutely และ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \ldots$$

Ex $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$ conv. (Ex. 4.12.1)

ใน Ex 4.13.2 เห็นว่า $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ conv. conditionally

สมมติให้ $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges ไปหา $S$

$$(\ln 2) = S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots \quad \text{—— } (1)$$

คูณ (1) ด้วย $\frac{1}{2}$

$$\frac{1}{2}S = \frac{1}{2}-\left(\frac{1}{2}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{1}{3}\right)-\left(\frac{1}{2}\cdot\frac{1}{4}\right)+\left(\frac{1}{2}\cdot\frac{1}{5}\right)-\left(\frac{1}{2}\cdot\frac{1}{6}\right)+\dots$$

$$\frac{1}{2}S = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\dots \quad \text{—— } (2)$$

$$\frac{1}{2}S = 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+0+\frac{1}{10}+0-\frac{1}{12}+\dots \quad \text{—— } (3)$$

(1)+(3);

$$S+\frac{1}{2}S = 1-\cancel{\frac{1}{2}}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cancel{\frac{1}{6}}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cancel{\frac{1}{10}}+\dots$$

$$+\ 0+\cancel{\frac{1}{2}}+0-\frac{1}{4}+0+\cancel{\frac{1}{6}}+0-\frac{1}{8}+0+\cancel{\frac{1}{10}}+\dots$$

$(S=\ln 2)$

$$= 1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\dots$$

$$\frac{3S}{2} = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\dots$$

$$\therefore\ \underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots = \ln 2 < \frac{3}{2}\ln 2 = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\dots}$$

หากอนุกรมไม่เป็นบวก, ไม่ converge แบบ absolutely (conv. มีเงื่อนไข)
จ.ทำให้เกิด rearrangement ของอนุกรมได้ผลบวกตัวอื่นที่ต่างออกไป
"Riemann's rearrangement Theorem"

Ex

$$1-1+1-1+1-1+1 = \frac{1}{2}$$
$$1+2+4+8+16+\dots = -1$$
$$1+2+3+4+5+\dots = -\frac{1}{2}$$

does it make sense?