ทบทวน <u>Alternating series</u> $\sum_{n=1}^{\infty}(-1)^{n-1}a_n \quad (a_n \geq 0)$

<u>Leibniz's Thm.</u> Alternating series converges if
(1) $a_n \geq a_{n+1}$
(2) $\lim_{n\to\infty} a_n = 0$

Convergence แบบต่างๆ

- $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.
  $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges
- $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} |a_n|$ div. but $\sum_{n=1}^{\infty} a_n$ converges.

$\sum_{n=1}^{\infty} a_n$

$\boxed{1}$ $\sum_{n=1}^{\infty} |a_n|$ conv.? $\xrightarrow{Y}$ conv. absolutely
$\downarrow N$
$\boxed{2}$ $\sum_{n=1}^{\infty} a_n$ conv.? $\xrightarrow{Y}$ conv. cond.
$\downarrow N$
diverges

<u>Ratio test for Absolute Convergence.</u>

$\sum_{n=1}^{\infty} a_n$, $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ Then
$$\begin{cases} \rho < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv. absolutely} \\ \rho > 1 \text{ or } \rho \to \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.} \\ \rho = 1 \Rightarrow \text{cannot be concluded!} \end{cases}$$

<u>ฟังก์ชันอนุกรมของตัวแปร</u> ให้ $x \in \mathbb{R}$

$\frac{a_1}{1-r}$

$$\underbrace{1 + x + x^2 + x^3 + \ldots + x^n + \ldots}_{\text{อนุกรม } \sum_{k=0}^{\infty} x^k} = \underbrace{\frac{1}{1-x}}_{f(x) = \frac{1}{1-x}} \quad (\text{ถ้า } |x| < 1) \quad [\,|x| < 1\,]$$

เป็นอนุกรมที่แทน $\frac{1}{1-x}$ เมื่อ $|x| < 1$

อนุกรมกำลังในพจน์ของ x

4.6 Power series in x

If $c_0, c_1, \ldots$ are constants and x is a variable, then the series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k + \ldots$$

is called a power series in x

Some examples are

1. $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$

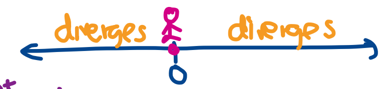
2. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$

4.6.1 Radius and Interval of Convergence

Theorem : For any power series $\sum_{k=0}^{\infty} c_k x^k$ exactly one of the following is true:

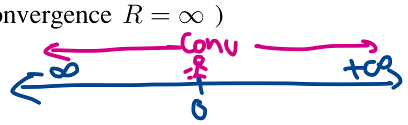
1. The series converges only for $x = 0$. (Radius of convergence $R = 0$)

Case ที่ $x=0$
แทน $x=0$
$\sum_{k=0}^{\infty} c_k x^k = 0+0+0+0+\ldots+0+\ldots$



Interval of Convergence $= \{0\}$

2. The series converges absolutely (and hence converges) for all real values of x. (Radius of convergence $R = \infty$)



Interval of convergence $= (-\infty, \infty) / \mathbb{R}$

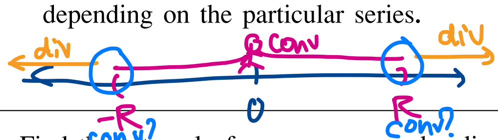
3. (a) The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$ (Radius of convergence $=R$) and;

   (b) diverges if $x < -R$ or $x > R$ and; $|x| < R \Leftrightarrow -R < x < R$

   (c) for $x = \pm R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Interval of Convergence $(-R, R)$
แต่ต้องตรวจสอบที่ขอบก่อนนะ

Ex
$[-R, R]$
$(-R, R]$
$[-R, R)$
$(-R, R)$



$|\Delta| \le \heartsuit \Leftrightarrow -\heartsuit \le \Delta \le \heartsuit$

$|\Delta| \ge \heartsuit \Leftrightarrow \Delta \ge \heartsuit$ or $\Delta \le -\heartsuit$

Ex. 4.15. Find the interval of convergence and radius of convergence of the following power series.

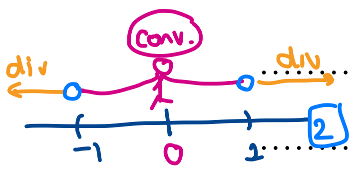
1. $\sum_{k=0}^{\infty} x^k$ $= 1 + x + x^2 + x^3 + x^4 + \ldots$ ($u_k = x^k$)

$\boxed{1}$ หาช่วงของการลู่เข้า (หาว่า x เป็นเท่าใดจึงจะทำให้อนุกรมลู่เข้า)
โดยพิจารณาจาก ratio test for absolute convergence

$\rho = \lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty} |x| = |x|$

$\therefore \rho = |x|$ converges เมื่อ $|x| = \rho < 1 \Leftrightarrow -1 < x < 1$

diverges เมื่อ $|x| = \rho > 1 \Leftrightarrow x < -1$ or $x > 1$



$\boxed{2}$ check ที่ขอบ $x = 1$: $\sum_{k=0}^{\infty} 1^k = 1+1+1+\ldots+1+\ldots$ div $\left(\lim_{n\to\infty} 1 = 1 \neq 0\right)$

$x = -1$: $\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+1-1+\ldots$ div $\left(\lim_{n\to\infty} (-1)^n \text{ DNE}\right)$

$\therefore$ Interval of convergence คือ $(-1, 1)$
radius of convergence คือ 1 #

2. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ u_k

$\boxed{1}$ พิจารณา $\displaystyle \rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!} \Big/ \frac{x^k}{k!}\right|$

$\displaystyle = \lim_{k\to\infty}\left|\frac{x\cdot\cancel{x^k}}{(k+1)\cancel{k!}}\cdot\frac{\cancel{k!}}{\cancel{x^k}}\right|$

$\displaystyle = \lim_{k\to\infty}\left|\frac{x}{k+1}\right|$

$= 0 < 1$

นั่นคือ $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$ converges absolutely for all $x \in \mathbb{R}$

$\therefore$ Interval of convergence $= (-\infty, \infty)$

Radius of convergence $= \infty$ #

3. $\displaystyle\sum_{k=0}^{\infty} k!x^k$

$\boxed{1}$ พิจารณา $\displaystyle \rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!\,x^{k+1}}{k!\,x^k}\right|$

$\displaystyle = \lim_{k\to\infty}\left|\frac{(k+1)\cancel{k!}\,x\cdot\cancel{x^k}}{\cancel{k!}\,\cancel{x^k}}\right|$

$\displaystyle = \lim_{k\to\infty}|(k+1)(x)|$

$\displaystyle = \lim_{k\to\infty}(k+1)|x| = \begin{cases} 0 & ;\ \text{if } x = 0 \\ +\infty & ;\ \text{if } x \neq 0 \end{cases}$

$\therefore \displaystyle\sum_{k=0}^{\infty} k!\,x^k$ converges absolutely if $x = 0$

and diverges if $x \neq 0$.

$\therefore$ Interval of convergence $= \{0\}$

Radius of convergence $= 0$ #

4.7 Power series in $x-a$

If $c_0, c_1, \ldots$ are constants and if we replace x by $(x-a)$, then the series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_k(x-a)^k + \ldots$$

is called a power series in $x-a$

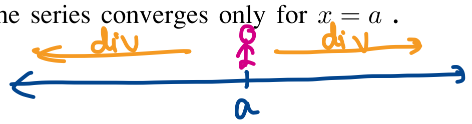
Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \ldots \qquad a=1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(x+3)^k}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \ldots \qquad a=-3$$

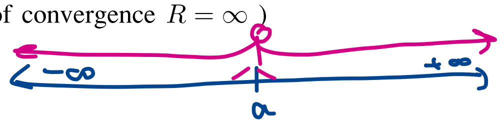
Theorem : For any power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ exactly one of the following is true:

1. The series converges only for $x=a$. (Radius of convergence $R=0$)



Interval of convergence $= \{a\}$

2. The series converges absolutely (and hence converges) for all real values of x. (Radius of convergence $R=\infty$)

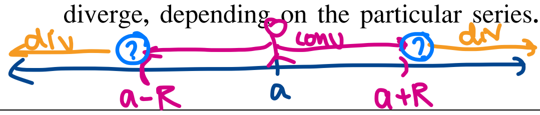


Interval of convergence $= (-\infty, \infty)$

3. (a) The series converges absolutely (and hence converges) for all x in some finite open interval $(a-R, a+R)$ (Radius of convergence $=R$) and;

   (b) diverges if $x < a-R$ or $x > a+R$ and;

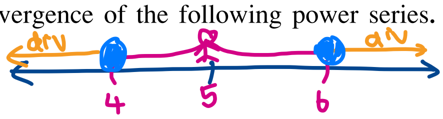
   (c) for $x = a \pm R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.



Interval of convergence $= (a-R, a+R)$

Ex. 4.16. Find the interval of convergence and radius of convergence of the following power series.

1. $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$



$\boxed{1}$ $\rho = \lim_{k\to\infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \Big/ \frac{(x-5)^k}{k^2} \right| = \lim_{k\to\infty} \left| \frac{(x-5)\cdot\cancel{(x-5)^k}}{(k+1)^2} \cdot \frac{k^2}{\cancel{(x-5)^k}} \right|$

$= \lim_{k\to\infty} \frac{k^2}{(k+1)^2}|x-5| = |x-5|$

• $\rho = |x-5| < 1$; then $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k^2}$ conv. absolutely.

$|x-5|<1$
$\Updownarrow$
$-1 < x-5 < 1$
$\therefore 4 < x < 6$

$\rho = |x-5| > 1$, then $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k^2}$ diverges

$|x-5|>1$
$x-5>1$ or $x-5<-1$
$\therefore x>6$ or $x<4$

$\boxed{2}$ เราลองดูขอบของช่วง

$\underline{x=4}$ ✓ $\sum_{k=1}^{\infty} \frac{(4-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ → พิจารณา $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ เป็น p-series $p=2>1$ ⇓ conv.

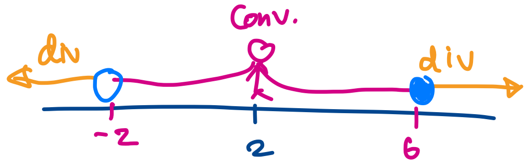
$\therefore \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ conv. absolutely.

$\underline{x=6}$ $\sum_{k=1}^{\infty} \frac{(6-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ → เป็นอนุกรม p ซึ่ง $p=2<1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}$ conv.

$\therefore$ Interval of convergence $= [4,6]$, radius of convergence $=1$ #

$|\Delta| = \heartsuit \Rightarrow \Delta = \heartsuit$ หรือ $\Delta = -\heartsuit$



2. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{(x-2)^k}{4^k\sqrt{k}}$

$\boxed{1}$ $\displaystyle\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x-2)^{k+1}}{4^{k+1}\sqrt{k+1}} \Big/ (-1)^k\frac{(x-2)^k}{4^k\sqrt{k}}\right|$

$\displaystyle= \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}\,\cancel{(x-2)^k}\cdot(x-2)\cdot\cancel{4^k}\sqrt{k}}{\cancel{(-1)^k}\,\cancel{4^k}\cdot 4\cdot\sqrt{k+1}\,\cancel{(x-2)^k}}\right|$

$\displaystyle= \lim_{k\to\infty}\left|\frac{(x-2)\sqrt{k}}{4\sqrt{k+1}}\right| = \lim_{k\to\infty}\sqrt{\frac{k}{k+1}}\left|\frac{x-2}{4}\right| = \left|\frac{x-2}{4}\right|$

$|x-2| < 4$

$\therefore -4 < x-2 < 4$

$-2 < x < 6$

$|x-2| > 4$

$x-2 > 4$ หรือ $x-2 < -4$

$\therefore x > 6$ หรือ $x < -2$

If $\left|\frac{x-2}{4}\right| < 1 \Leftrightarrow |x-2| < 4$, then the series converges absolutely

If $\left|\frac{x-2}{4}\right| > 1 \Leftrightarrow |x-2| > 4$, then diverges

$\boxed{2}$ จุดขอบ set $\left|\frac{x-2}{4}\right| = 1 \Leftrightarrow |x-2| = 4 \Leftrightarrow x-2 = 4$ or $x-2 = -4$

$\therefore x = 6$ or $x = -2$

จุดขอบ $\boxed{x=6}$: $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\cancel{(6-2)^k}}{\cancel{4^k}\sqrt{k}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}} \longrightarrow$ ① $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^k}{\sqrt{k}}\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ p-series $p=\frac{1}{2}<1$ div

② Leibniz's Thm

check $\boxed{1}$ $a_n \geq a_{n+1}$ เพราะ $\sqrt{n+1} \geq \sqrt{n}$ $\forall n \geq 1$

ทด $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} \Leftrightarrow \sqrt{n+1} \geq \sqrt{n}$

$\therefore a_n = \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} = a_{n+1}$ ✓

$\boxed{2}$ $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

$\therefore$ ได้ว่า $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ converges

$\boxed{x=-2}$ ✗ $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k(-4)^k}{4^k\sqrt{k}} = \sum_{k=1}^{\infty}\frac{\cancel{(4)^k}}{\cancel{4^k}\sqrt{k}} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ div

$\therefore$ Interval of convergence $= (-2, 6]$

Radius of convergence $= 4$ #

HW $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$

Exercise: Find the interval of convergence and radius of convergence of the following power series.

1. $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k+1}$

2. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}$

3. $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{1+k^2}$

4. $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{2^k}(x-5)^k$

5. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k(x+5)^k$

6. $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x+1)^k}{k}$