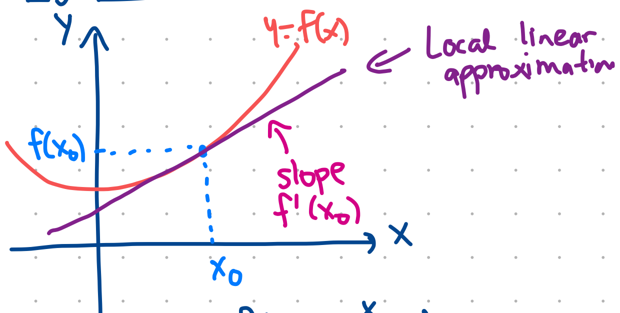
ทบทวน Power series

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_k x^k + \dots$$

$$\underbrace{\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_k(x-a)^k + \dots}_{\text{หา interval of convergence / radius of convergence}}$$

Taylor Series / Maclaurin Series



$$f(x) \approx P_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\therefore f(x) \approx P_2(x) = c_0 + c_1 x + c_2 x^2$$

สมบัติที่ P_2 ควรจะมี

1. $f(x_0) = P_2(x_0)$
2. $f'(x_0) = P_2'(x_0)$
3. $f''(x_0) = P_2''(x_0)$

Ex $f(x) = e^x$ ประมาณด้วยพหุนามดีกรีสอง ใกล้ๆ จุด $x_0 = 0$

ให้ $P_2(x) = c_0 + c_1 x + c_2 x^2$

$P_2'(x) = c_1 + 2c_2 x$

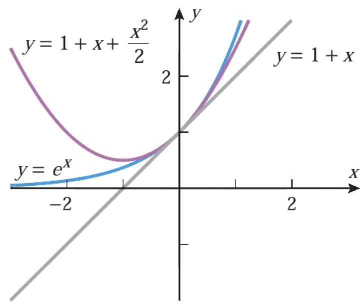
$P_2''(x) = 2c_2$

$$\therefore \quad f(x_0) = P_2(x_0) \iff f(0) = 1 = c_0 \qquad \therefore c_0 = 1$$

$$f'(x_0) = P_2'(x_0) \iff f'(0) = 1 = c_1 \qquad \therefore c_1 = 1$$

$$f''(x_0) = P_2''(x_0) \iff f''(0) = 1 = 2c_2 \qquad \therefore c_2 = \frac{1}{2}$$

$$\therefore P_2(x) = 1 + x + \frac{1}{2}x^2$$



สมมติ $f(x)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้ถึงอันดับ n ที่ $x_0 = 0$

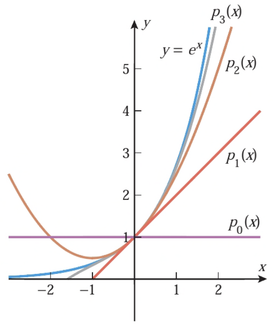
$$f(x) \approx P_n(x) = \underbrace{c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n}_{}$$

หา $c_0, c_1, \dots, c_n$ ที่มีสมบัติ

1. $f(0) = P_n(0)$
2. $f'(0) = P_n'(0)$
3. $f''(0) = P_n''(0)$

$\vdots$

$f^{(n)}(0) = P_n^{(n)}(0)$



$$P_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots + n c_n x^{n-1}$$

$$P_n''(x) = 2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 c_4 x^2 + \dots + n(n-1)c_n x^{n-2}$$

$$P_n'''(x) = 3\cdot 2 c_3 + 4\cdot 3\cdot 2 c_4 x + \dots + n(n-1)(n-2)c_n x^{n-3}$$

$f(0) = c_0$

$f'(0) = c_1$

$f''(0) = 2c_2 \iff c_2 = \frac{f''(0)}{2!}$

$f'''(0) = 3!\,c_3 \iff c_3 = \frac{f'''(0)}{3!}$

$\vdots$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$\therefore f(x) \approx \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n}_{\text{Maclaurin Polynomial of degree } n.}$$

Maclaurin Series : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

f หาอนุพันธ์ได้ทุกอันดับ ที่ $x_0 = 0$

Taylor series : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$

f หาอนุพันธ์ได้ทุกอันดับ ที่ $x_0 = a$

4.8 Taylor and Maclaurin Series

Definition : If $f(x)$ has derivaives of all orders at a , then we call the series

$$\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

the **Taylor series** for f at a.
In the special case where $a=0$, this series becomes

$$\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

in which case we call it the **Maclaurin series** for f.

Ex. 4.17. Find the Taylor series for

1. $f(x)=3x^3-5x^2+x+2$ about $a=1$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n = f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\ldots$$

$f(x)=3x^3-5x^2+x+2$; $f(1)=3-5+1+2=1$

$f'(x)=9x^2-10x+1$; $f'(1)=9-10+1=0$

$f''(x)=18x-10$; $f''(1)=18-10=8$

$f'''(x)=18$; $f'''(1)=18$

$f^{(4)}(x)=0$; $\forall n \geq 4$ ได้ว่า $f^{(n)}(1)=0$

$\therefore f(x)=1+0+\frac{8}{2!}(x-1)^2+\frac{18}{3!}(x-1)^3$

2. $f(x)=\dfrac{1}{x}$ about $a=-1$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n = f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\ldots$$

$f(x)=\frac{1}{x}$; $f(-1)=-1$

$f'(x)=-\frac{1}{x^2}$; $f'(-1)=-1$

$f''(x)=\frac{2}{x^3}$; $f''(-1)=-2$

$f'''(x)=-\frac{3\cdot 2}{x^4}=-\frac{3!}{x^4}$; $f'''(-1)=-3!$

$f^{(4)}(x)=\frac{4!}{x^5}$; $f^{(4)}(-1)=-4!$

$\vdots$

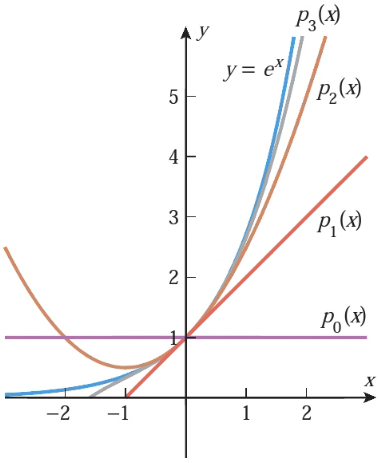
$f^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}}$; $f^{(n)}(-1)=-n!$

$$\therefore f(x) = -1-1(x+1)-\frac{\cancel{2!}}{\cancel{2!}}(x+1)^2-\frac{\cancel{3!}}{\cancel{3!}}(x+1)^3-\ldots-\frac{\cancel{n!}}{\cancel{n!}}(x+1)^n-\ldots$$
$$= -1-(x+1)-(x+1)^2-(x+1)^3-\ldots-(x+1)^n-\ldots$$
$$= \sum_{n=0}^{\infty}-(x+1)^n$$

Ex. 4.18. Find the Maclaurin series for $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

1. $f(x) = e^x$

$e^{0.01} = f(0.01) = 1 + (0.01) + \frac{1}{2!}(0.01)^2 + \frac{1}{3!}(0.01)^3 + \dots$



$f(x) = e^x \quad ; f(0) = 1$

$f'(x) = e^x \quad ; f'(0) = 1$

$f''(x) = e^x \quad ; f''(0) = 1$

$f'''(x) = e^x \quad ; f'''(0) = 1$

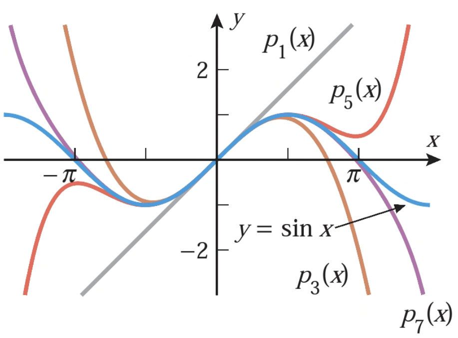
$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \quad ; f^{(n)}(0) = 1$

$\therefore f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $f(x) = \sin x$ $\qquad f(0) = 0$



$f'(x) = \cos x \quad ; f'(0) = 1$

$f''(x) = -\sin x \quad ; f''(0) = 0$

$f'''(x) = -\cos x \quad ; f'''(0) = -1$

$f^{(4)}(x) = \sin x \quad ; f^{(4)}(0) = 0$

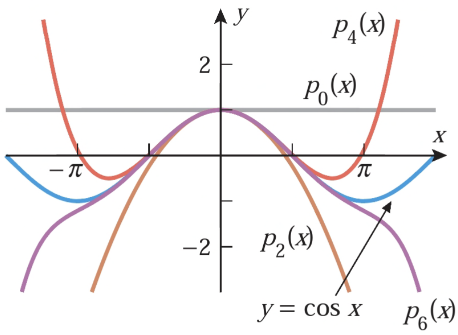
$\vdots \qquad \vdots$

$\therefore f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$f^{(n)}(0) = \begin{cases} 0 & ; n \text{ even} \\ 1 & ; n = 1, 5, 9, \dots \\ -1 & ; n = 3, 7, 11, \dots \end{cases}$

3. $f(x) = \cos x$ $\qquad f(0) = 1$



$f'(x) = -\sin x \quad ; f'(0) = 0$

$f''(x) = -\cos x \quad ; f''(0) = -1$

$f'''(x) = \sin x \quad ; f'''(0) = 0$

$f^{(4)}(x) = \cos x \quad ; f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin x \quad ; f^{(5)}(0) = 0$

$\vdots$

$f(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$f^{(n)}(0) = \begin{cases} 0 & ; n \text{ odd} \\ 1 & ; n = 0, 4, 8, 12, \dots \\ -1 & ; n = 2, 6, 10, \dots \end{cases}$

$\cos 1^\circ = \cos \frac{\pi}{180}$

$\cos\left(\frac{\pi}{180}\right) = f\left(\frac{\pi}{180}\right) = 1 - \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{180}\right)^2 - \frac{1}{6!}\left(\frac{\pi}{180}\right)^6 + \dots$

4. $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$ $\quad f(0) = 1$

$f'(x) = -\dfrac{1}{(1-x)^2}(-1)$

$= \dfrac{1}{(1-x)^2}$

$= (1-x)^{-2}$

$f''(x) = -2(1-x)^{-3}(-1)$

$= 2(1-x)^{-3}$

$= \dfrac{2}{(1-x)^3}$

$f'(x) = \dfrac{1}{(1-x)^2}$; $f'(0) = +1$

$f''(x) = \dfrac{2}{(1-x)^3}$; $f''(0) = 2$

$f'''(x) = \dfrac{3!}{(1-x)^4}$; $f'''(0) = 3!$

$f^{(4)}(x) = \dfrac{4!}{(1-x)^5}$; $f^{(4)}(0) = 4!$

$\vdots$ $\qquad\qquad \vdots$

$f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$; $f^{(n)}(0) = n!$

$$f(x) = 1 + x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \dots + \frac{n!}{n!}x^n + \dots$$
$$= 1 + x + x^2 + \dots + x^n$$
$$= \sum_{n=0}^{\infty} x^n \quad \text{เมื่อ } |x| < 1$$

Ex $f(x) = \ln(1+x)$ รอบ $x = 0$

$f(x) = \ln(1+x)$; $f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$; $f'(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2}$; $f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3}$; $f'''(0) = 2$

$f^{(4)}(x) = -\dfrac{3!}{(1+x)^4}$; $f^{(4)}(0) = -3!$

$\vdots$ $\qquad\qquad \vdots$

$f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$; $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$$\boxed{\begin{aligned} f(x) &= \ln(1+x) \\ &= 0 + 1\cdot x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \end{aligned}}$$

$f(x) = \ln(1+x)$

จาก $\ln(2) = f(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$

Ex กระจายอนุกรมของ $f(x) = x\ln(1+x)$

จาก $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$

$$xf(x) = x\ln(1+x) = x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right)$$
$$= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots$$

* บวก/แทนค่า/คูณด้วยค่าคงตัว/คูณด้วย x^n ได้อนุกรม Taylor/Maclaurin.

Ex $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

Ex ตย. Maclaurin Series

① $\frac{1}{1-x} = 1+x+x^2+\dots+x^n+\dots = \sum_{n=0}^{\infty} x^n$ เมื่อ $|x|<1$

② $\frac{1}{1+x} = 1-x+x^2-\dots+(-x)^n+\dots = \sum_{n=0}^{\infty} (-x)^n$ เมื่อ $|x|<1$

③ $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\forall x \in \mathbb{R}$

④ $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\forall x \in \mathbb{R}$

⑤ $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}+\dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\forall x \in \mathbb{R}$

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⑥ $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+\dots+(-1)^n\frac{x^n}{n}+\dots = \sum_{n=0}^{\infty} (-1)^{n-1}\frac{x^n}{n}$; $-1 < x \leq 1$

⑦ $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots+(-1)^n\frac{x^{2n+1}}{2n+1}+\dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$; $|x| \leq 1$